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Convergence in Almost Periodic Competition Diffusion Systems

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The paper deals with the convergence of positive solutions for almost-periodic competition diffusion systems. The asymptotic almost periodicity of a positive solution for such a system is described by the almost periodicity of the ω-limit set of the corresponding positive motion in the associated skew-product flow. In the framework of the skew-product flow, it will be proved that the ω-limit set of any spatially homogeneous positive motion contains at most two minimal sets which are both almost automorphic. It will also be proved that if each spatially homogeneous positive solution is asymptotically almost periodic and each spatially homogeneous positive almost periodic solution is lower (upper) asymptotically Lyapunov stable, then every positive solution converges to a spatially homogeneous almost periodic solution. Several important special cases are described where every positive solution converges to a spatially homogeneous almost-periodic solution.

1. INTRODUCTION

This paper is devoted to the study of the reaction diffusion system

$$\begin{aligned} u_t &= k_1(t,x)\Delta u + uf_1(t,u,v) \\ v_t &= k_2(t,x)\Delta v + vf_2(t,u,v) \end{aligned} \tag{1.1}$$

in some cylindrical time-space domain $(0,\infty)\times\Omega\subset\mathbb{R}^{N+1}$, together with zero Neumann boundary data on the lateral boundary $(0,\infty)\times\partial\Omega$, and the corresponding system of ordinary differential equations

$$\begin{aligned} u_t &= uf_1(t,u,v) \\ v_t &= vf_2(t,u,v), \end{aligned} \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, k_1, k_2, f_1, and f_2 are C^2 functions and are uniformly almost periodic in t (see Section 2), and

$$k_i(t,x) \geq \delta_0 \qquad \text{for } t \in \mathbb{R} \text{ and } x \in \overline{\Omega}, \tag{1.3}$$

$$\frac{\partial f_i}{\partial u}, \frac{\partial f_i}{\partial v} \leq -\delta_0 < 0 \qquad \text{for } t, u, v \in \mathbb{R}, \tag{1.4}$$

$$\lambda_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T f_i(t,0,0)\, dt > 0, \tag{1.5}$$

$$f_i(t,u,v) \leq -\delta_0 \quad \text{for } t \in \mathbb{R} \text{ and } u \geq 0,\ v \gg 1 \text{ or } u \gg 1,\ v \geq 0, \tag{1.6}$$

where $\delta_0 > 0$ and $i = 1,2$. Note that solutions of (1.2) are spatially homogeneous solutions of (1.1).

In mathematical ecology, the system (1.1) or (1.2) models the competition between two species. The assumption of the almost periodicity of k_1, k_2, f_1, and f_2 is a way of incorporating the time-dependent variability of the environment, especially when the various components of the environment are periodic with not necessarily commensurate periods.

Ecologically, it is of interest to consider the long-term behavior of positive solutions of the systems (1.1) and (1.2). A solution $(u(t,x), v(t,x))$ of (1.1) is said to be *positive* (*strictly positive*) if $u(t,x)$, $v(t,x) \geq 0$ (> 0) and $(u(x,t), v(x,t)) \not\equiv (0,0)$ for $x \in \overline{\Omega}$ and t in the interval of existence. A solution $(u(t), v(t))$ of (1.2) is said to be *positive* (*strictly positive*) if $u(t), v(t) \geq 0$ (> 0) and $(u(t), v(t)) \not\equiv (0,0)$ for t in the interval of existence. The above problem has been investigated in numerous papers, see [5, 11, 15, 16, 18, 19, 30], etc., for the time-independent case, [3, 4, 7, 8, 13, 14, 17, 25], etc., for the time-periodic case, and [1, 2, 9, 10, 20], etc., for the time-almost-periodic or general-time-dependent case.

In the time-independent case, one obtains that if (1.2) has one globally stable positive equilibrium or has a certain continuous family of positive equilibria, then any positive solution of (1.1) converges to an equilibrium of (1.2) (see [19], etc., for the former case and [5] for the later one). In the time-periodic case, if (1.2) has one globally stable positive periodic solution, it has also been proved that any positive solution of (1.1) converges to a positive periodic solution of (1.2) (see [3, 4, 8, 13, 14, 17], etc.). In [25], the author studied the convergence of positive solutions of (1.1) when k_i, f_i ($i = 1,2$) are periodic and the positive periodic solutions of (1.1) form a totally ordered set with certain stability properties (see [25] for details).

In this paper, we shall study the long-term behavior of positive solutions of (1.1) and (1.2). Suppose that $(u^*(t,x), v^*(t,x))$ ($(u^*(t), v^*(t))$) is a solution of (1.1) ((1.2)). We say a solution $(u(t,x), v(t,x))$ ($(u(t), v(t))$) *con-*

verges to $(u^*(t,x), v^*(t,x))$ $((u^*(t), v^*(t)))$ if

$$(u(t,x), v(t,x)) - (u^*(t,x), v^*(t,x)) \to (0,0)$$
$$(((u(t), v(t)) - (u^*(t), v^*(t)) \to (0,0))$$

in an appropriate norm as $t \to \infty$. Moreover, we say $(u(t,x), v(t,x))$ $((u(t), v(t)))$ is *asymptotically almost periodic* if $(u^*(t,x), v^*(t,x))$ $((u^*(t), v^*(t)))$ is almost periodic in t.

In order to carry out our study, we first define a dynamical system associated with (1.1) in the following way. Let $k(t,x) = (k_1(t,x), k_2(t,x))$, $f(t,u,v) = (f_1(t,u,v), f_2(t,u,v))$, $H(k,f)$ be the hull of (k,f), and $(H(k,f), \mathbb{R})$ be the time-translation flow: $((l,g),t) = (l \cdot t, g \cdot t)$ for $(l,g) \in H(k,f)$ (see Section 2). Let $X \subset L^p(\Omega)$ $(p > n)$ be a fractional power space of $-\Delta$: $\mathscr{D} \to L^p(\Omega)$ satisfying $X \hookrightarrow C^1(\overline{\Omega})$, where $\mathscr{D} = \{u \in H^{2,p}(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$ (see [12]). Now let

$$\Pi_t\colon X \times X \times H(k,f) \to X \times X \times H(k,f) \tag{1.7}$$

be defined by

$$\Pi_t(u_0, v_0, l, g) = (u(t,\cdot\,; u_0, v_0, l, g), v(t,\cdot\,; u_0, v_0, l, g), l \cdot t, g \cdot t),$$

where $l = (l_1, l_2)$, $g = (g_1, g_2)$, $(l,g) \in H(k,f)$, and $(u(t,x; u_0, v_0, l, g)$, $v(t,x; u_0, v_0, l, g))$ is the solution of

$$\begin{aligned} u_t &= l_1(t,x)\Delta u + u g_1(t,u,v) \\ v_t &= l_2(t,x)\Delta v + v g_2(t,u,v) \end{aligned} \tag*{$(1.8)_{(l,g)}$}$$

in the time-space domain $(0,\infty) \times \Omega \subset \mathbb{R}^{N+1}$, together with zero Neumann boundary data on the lateral boundary $(0,\infty) \times \partial\Omega$ and initial condition $(u(0,x; u_0, l, g), v(0,x; u_0, v_0, l, g)) = (u_0(x), v_0(x))$. Then Π_t is a (local) semiflow on $X \times X \times H(k,f)$ which one usually calls the *skew-product flow* generated by (1.1).

Observe that in the case that (u_0, v_0) is spatially homogeneous, $(u(t,\cdot\,; u_0, v_0, l, g), v(t,\cdot\,; u_0, v_0, l, g))$ is also spatially homogeneous and does not dependent on l. We then write $(u(t; u_0, v_0, g), v(t; u_0, v_0, g))$ for simplicity and note that $t \mapsto (u(t; u_0, v_0, g), v(t; u_0, v_0, g))$ is the solution of

$$\begin{aligned} u_t &= u g_1(t,u,v) \\ v_t &= v g_2(t,u,v) \end{aligned} \tag*{$(1.9)_g$}$$

with $(u(0; u_0, v_0, g), v(0; u_0, v_0, g)) = (u_0, v_0)$. In the following, $(u(t,\cdot\,; u_0, v_0, l, g), v(t,\cdot\,; u_0, v_0, l, g))$ denotes the solution of $(1.8)_{(l,g)}$ and $(u(t; u_0, v_0, g), v(t; u_0, v_0, g))$ denotes either the solution of $(1.9)_g$ or the solution of $(1.8)_{(l,g)}$ with $(l, g) \in H(k, f)$. We define

$$\pi_t: \mathbb{R} \times \mathbb{R} \times H(f) \to \mathbb{R} \times \mathbb{R} \times H(f), \tag{1.10}$$
$$\pi_t(u_0, v_0, g) = \big(u(t; u_0, v_0, g), v(t; u_0, v_0, g), g \cdot t\big)$$

to be the skew-product flow associated to (1.2).

In terms of the skew-product flow (1.7) ((1.10)), the study of the long-term behavior of a solution pair $(u(t, x; u_0, v_0, k, f),\ v(t, x; u_0, v_0, k, f))$ $((u(t; u_0, v_0, f),\ v(t; u_0, v_0)))$ of (1.1) ((1.2)) gives rise to the problem of understanding the ω-limit set $\omega(u_0, v_0, k, f)$ $(\omega(u_0, v_0, f))$ of the motion $\Pi_t(u_0, v_0, k, f)$ $(\pi_t(u_0, v_0, f))$. Clearly, if $(u(t; u_0, v_0, f), v(t; u_0, v_0, f))$ is bounded in $\mathbb{R} \times \mathbb{R}$ for t in the existence interval of the solution, then it is globally defined, and $\omega(u_0, v_0, f)$ is a nonempty connected compact subset of $\mathbb{R} \times \mathbb{R} \times H(f)$. By regularity and a priori estimates for parabolic equations, if $(u(t, \cdot\,; u_0, v_0, k, f), v(t, \cdot\,; u_0, v_0, k, f))$ is bounded in $X \times X$ for t in the existence interval of the solution, then $(u(t, \cdot\,; u_0, v_0, k, f), v(t, \cdot\,; u_0, v_0, k, f))$ is a globally defined classical solution; moreover, for any $\delta > 0$, $\{(u(t, \cdot\,; u_0, v_0, k, f), v(t, \cdot\,; u_0, v_0, k, f)) \mid t \geq \delta\}$ is relatively compact in $X \times X$. Therefore, $\omega(u_0, v_0, k, f)$ is a nonempty connected compact subset of $X \times X \times H(k, f)$. Furthermore, since Π_t has a unique continuous backward time extension on the ω-limit set $\omega(u_0, v_0, k, f)$, it defines the usual skew-product (two-sided) flow on $\omega(u_0, v_0, k, f)$. Now the asymptotic almost periodicity of $(u(t, \cdot\,; u_0, v_0, k, f), v(t, \cdot\,; u_0, v_0, k, f))$ is determined by the almost periodicity of $\omega(u_0, v_0, k, f)$. For example, if $\omega(u_0, v_0, k, f)$ is a 1-cover of $H(k, f)$ (hence almost periodic minimal) (see Section 2 for definition), then $(u(t, \cdot\,; u^*, v^*, k, f), v(t, \cdot\,; u^*, v^*, k, f))$ is almost periodic in t and

$$\begin{aligned}\|&\big(u(t, \cdot\,; u_0, v_0, k, f), v(t, \cdot\,; u_0, v_0, k, f)\big)\\ &\quad - \big(u(t, \cdot\,; u^*, v^*, k, f), v(t, \cdot\,; u^*, v^*, k, f)\big)\| \to 0\end{aligned}$$

as $t \to \infty$, where $(u^*, v^*, k, f) \in \omega(u_0, v_0, k, f)$ and $\|\cdot\|$ denotes the norm in $X \times X$.

Throughout this paper, Π_t and π_t represent the skew-product flows (1.7) and (1.10), respectively. X^+ and $\mathbb{R}^+$ denote the positive cones of X and $\mathbb{R}$, respectively. P is the natural projection

$$P: X \times X \times H(k, f) \to H(k, f), \qquad P(u, v, l, g) = (l, g), \tag{1.11}$$

or

$$P: \mathbb{R} \times \mathbb{R} \times H(f) \to H(f), \qquad P(u, v, g) = g. \tag{1.12}$$

Note that it follows from [22] that, for any $g \in H(f)$, there are $u_g, v_g \in \mathrm{Int}(\mathbb{R}^+)$ such that $(u(t; u_g, 0, g), 0)$ and $(0, v(t; 0, v_g, g))$ are almost-periodic solutions of $(1.9)_g$ (see Section 3).

In the framework of the skew-product flow, we can state our main results as follows.

THEOREM A. *Consider* (1.2). *For any* $(u_0, v_0, g_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, $\omega(u_0, v_0, g_0)$ *is totally ordered with respect to the ordering* $\leq_2$ (*see Section* 3 *for the meaning of* $\leq_2$) *and contains at most two minimal sets, and any minimal set is an almost* 1-*cover of* $H(f)$ (*that is, an almost automorphic extension of* $H(f)$). *Hence, any positive almost-periodic solution* $(u(t), v(t))$ *of* (1.2) (*if it exists*) *is harmonic* (*that is,* $\mathscr{M}(u(\cdot), v(\cdot)) \subset \mathscr{M}(f)$, *where* $\mathscr{M}(\cdot)$ *denotes the frequency module of an almost-periodic function*) (*Theorem* 4.1).

THEOREM B. *Consider* (1.1). *If, for any* $(u_0, v_0, g) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, $\omega(u_0, v_0, g)$ *is a* 1-*cover of* $H(f)$ (*that is, any positive solution of* (1.2) *is asymptotically almost periodic*) *and any positive almost-periodic motion of* (1.10) *is lower* (*upper*) *asymptotically Lyapunov stable* (*see Definition* 5.1) *or* (1.10) *has a globally stable positive almost-periodic minimal set, then for any* $(u_0, v_0, l, g) \in Int(X^+) \times Int(X^+) \times H(k, f)$, $\omega(u_0, v_0, l, g)$ *is a* 1-*cover of* $H(k, f)$ *and is spatially homogeneous* (*hence any positive solution of* (1.1) *is asymptotically almost periodic and spatially homogeneous* (*Theorem* 5.1).

We mention [25–27] for convergence results under a Lyapunov stability hypothesis in periodic settings. We point out that the above theorems extend most existing results on the asymptotic dynamics of competition systems. For example, Theorem A implies that if k, f are periodic of period T, then for any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\omega(u_0, v_0, f)$ is a 1-cover of $H(f)$, and hence $(u(t; u_0, v_0, f), v(t; u_0, v_0))$ is asymptotically T-periodic (Corollary 4.3). Such results have been proved in [17]. Theorems A and B imply that if (1.2) has a certain continuous family of positive almost-periodic solutions or has one globally-stable positive almost-periodic solution, then any positive solution of (1.1) is asymptotically almost periodic and spatially homogeneous. In particular, we have

THEOREM C. *Consider* (1.1).

(1) *If* $f_1(t, u, v) = f_2(t, u, v) = f^*(t, u + v)$, *then* (1.2) *has a continuous family of positive almost-periodic solutions and, for any* $(u_0, v_0, l, g) \in Int(X^+) \times Int(X^+) \times H(k, f)$, $\omega(u_0, v_0, l, g)$ *is a* 1-*cover of* $H(k, f)$ *and is spatially homogeneous* (*Corollaries* 4.5 *and* 5.3).

(2) *If* $f_i(t, u, v) = a_i(t) - b_i(t)u - c_i(t)v$ $(i = 1, 2)$, *then the following holds*.

(i) *If* $a_{1L} > c_{1M} a_{2M} / c_{2L}$ *and* $a_{2L} > a_{1M} b_{2M} / b_{1L}$, *then there is a unique spatially homogeneous strictly positive almost-periodic minimal set*

$K^* = \{(u^*(g), v^*(g), l, g) \colon (l, g) \in H(k, f)\}$ *of* Π_t *such that, for any* $(u_0, v_0, l, g) \in Int(X^+) \times Int(X^+) \times H(k, f)$, $\|\Pi_t(u_0, v_0, l, g) - \Pi_t(u^*(g), v^*(g), l, g)\| \to 0$ *as* $t \to \infty$.

(ii) *If* $a_{1L} > c_{1M} a_{2M}/c_{2L}$ *and* $a_{2M} < b_{2L} a_{1L}/b_{1M}$, *then for any* $(u_0, v_0, l, g) \in Int(X^+) \times Int(X^+) \times H(k, f)$, $\|\Pi_t(u_0, v_0, l, g) - \Pi_t(u_g, 0, l, g)\| \to 0$ *as* $t \to \infty$.

(iii) *If* $a_{1M} < c_{1L} a_{2L}/c_{2M}$ *and* $a_{2L} > b_{2M} a_{1M}/b_{2L}$, *then for any* $(u_0, v_0, l, g) \in Int(X^+) \times Int(X^+) \times H(k, f)$, $\|\Pi_t(u_0, v_0, l, g) - \Pi_t(0, v_g, l, g)\| \to 0$ *as* $t \to \infty$.

where $a_{iM(L)} = sup(inf)_{t \in \mathbb{R}} a_i(t)$, $b_{iM(L)} = sup(inf)_{t \in \mathbb{R}} b_i(t)$, $c_{iM(L)} = sup(inf)_{t \in \mathbb{R}} c_i(t)$, *and* $a_{iL}, b_{iL}, c_{iL} > 0$ $(i = 1, 2)$ (*Corollaries* 4.6 *and* 5.4).

Note that in the periodic case, Theorem C(1) has been proved in [25], and Theorem C(2) has been proved in [3]. In the general nonautonomous case, the author of [1] proved that if the condition of Theorem C(2.i) holds, then for any $(u_1, v_1), (u_2, v_2) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+)$,

$$\begin{aligned}&\big(u(t; u_1, v_1, f), v(t; u_1, v_1, f)\big)\\ &\qquad - \big(u(t; u_2, v_2, f), v(t; u_2, v_2, f)\big) \to (0, 0) \qquad \text{as } t \to \infty.\end{aligned}$$

But [1] did not conclude that any spatially homogeneous positive solution is asymptotically almost periodic when f is almost periodic. In [9], the author proved Theorem C(2.i) under the additional conditions that $b_{1L} > c_{1M} + b_{2M} + c_{2M} + \mu$ and $c_{2L} > b_{1M} + c_{1M} + b_{2M} + \mu$ for some $\mu > 0$.

The paper is organized as follows. In Section 2, we are going to summarize basic properties of almost-periodic functions and flows which will be used in later sections. We present basic properties of (1.1) and (1.2) in Section 3. Section 4 is devoted to the investigation of positive solutions of (1.2) and Theorem A is proved in this section. We explore the asymptotic behavior of positive solutions of (1.1) and prove Theorem B and Theorem C in Section 5.

2. ALMOST PERIODIC FUNCTIONS AND FLOWS

In this section, we summarize some basic properties of almost-periodic functions for use in later sections. The reader is referred to [6, 29] for definitions and notations.

DEFINITION 2.1. (1) A function $f \in C(\mathbb{R}, \mathbb{R}^m)$ is said to be almost periodic if, for any $\epsilon > 0$, the set

$$T(\epsilon) = \{\tau : |f(t + \tau) - f(t)| < \epsilon,\ t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$.

(2) A function $f \in C(\mathbb{R} \times D, \mathbb{R}^m) : (t, x) \mapsto f(t, x)$ $(D \subset \mathbb{R}^n)$ is said to be uniformly almost periodic in t if f is almost periodic in t for each $x \in D$ and, for any compact set $E \subset D$, f is uniformly continuous on $\mathbb{R} \times E$.

(3) Let $f \in C(\mathbb{R} \times D, \mathbb{R}^m)$ $(D \subset \mathbb{R}^n)$ be uniformly almost periodic in t. Then $H(f) = cl\{f \cdot \tau : \tau \in \mathbb{R}\}$ is called the hull of f, where $f \cdot \tau(t, x) = f(t + \tau, x)$ and the closure is taken in the compact open topology.

(4) Let $f \in C(\mathbb{R} \times D, \mathbb{R}^m)$ $(D \subset \mathbb{R}^n)$ be uniformly almost periodic in t, and

$$f(t, x) \sim \sum_{\lambda \in \mathbb{R}} a_\lambda(x) e^{i\lambda t}$$

be its Fourier series, where

$$a_\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s, x) e^{-i\lambda s} \, ds.$$

Then $\mathscr{S}(f) = \{\lambda : a_\lambda(x) \not\equiv 0\}$ is called the Fourier spectrum of f, and $\mathscr{M}(f) =$ the smallest additive subgroup of $\mathbb{R}$ containing $\mathscr{S}(f)$ is called the frequency module of f.

It is known that both $\mathscr{S}(f)$ and $\mathscr{M}(f)$ are countable ([6]), and moreover the following holds.

LEMMA 2.1. (1) *A function $f \in C(\mathbb{R} \times D, \mathbb{R}^m)$ $(D \subset \mathbb{R}^n)$ is uniformly almost periodic in t if and only if, for any sequences $\{\alpha_n'\}, \{\beta_n'\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha_n'\}$, $\{\beta_n\} \subset \{\beta_n'\}$ such that*

$$\lim_{k \to \infty} \lim_{n \to \infty} f(t + \alpha_n + \beta_k, x) = \lim_{n \to \infty} f(t + \alpha_n + \beta_n, x)$$

uniformly for x in compact sets.

(2) *Let $f(t, x)$ and $g(t, y)$ ($f \in C(\mathbb{R} \times D_1, \mathbb{R}^m)$, $g \in C(\mathbb{R} \times D_2, \mathbb{R}^k)$, $D_1 \subset \mathbb{R}^n$, $D_2 \subset \mathbb{R}^l$) be two uniformly almost-periodic functions in t. Then $\mathscr{M}(g) \subset \mathscr{M}(f)$ if and only if, for any sequence $\{\alpha_n\} \subset \mathbb{R}$, if $\lim_{n \to \infty} f(t + \alpha_n, x) = f(t, x)$ uniformly for x in compact sets, then $\lim_{n \to \infty} g(t + \alpha_n, y) = g(t, y)$ uniformly for y in compact sets.*

Proof. See [6] or [29]. ∎

Throughout the remainder of this section, $(X, \mathbb{R})$ and $(Y, \mathbb{R})$ will denote dynamical systems with X, Y being compact metric spaces. Moreover, let $\alpha = \{t_n\}$ be a net in $\mathbb{R}$ and $x_0 \in X$, then we write $T_\alpha x_0$ for the limit of $x_0 \cdot t_n$ provided that such a limit exists.

DEFINITION 2.2. (1) A point $x_0 \in X$ is called an almost-periodic point if, for any nets α', β' in $\mathbb{R}$, there are subnets α, β such that

$$T_\alpha T_\beta x_0 \cdot t = T_{\alpha+\beta} x_0 \cdot t$$

uniformly in t $\in \mathbb{R}$.

(2) $(X, \mathbb{R})$ is almost-periodic minimal if it is minimal and contains an almost-periodic point.

DEFINITION 2.3. (1) A continuous map P: $X \to Y$ is said to be a flow homomorphism from $(X, \mathbb{R})$ to $(Y, \mathbb{R})$ if $P(X) = Y$, and $P(x \cdot t) = P(x) \cdot t$ for any $x \in X$ and $t \in \mathbb{R}$.

(2) X is called a 1-cover (or an almost-periodic extension) of Y if there is a flow homomorphism P: $X \to Y$ such that $P^{-1}(y)$ is a singleton for any $y \in Y$.

(3) X is called an almost 1-cover (or an almost automorphic extension) of Y if there is a flow homomorphism P: $X \to Y$ such that $P^{-1}(y)$ is a singleton for at least one $y \in Y$.

LEMMA 2.2. *If P: $X \to Y$ is a flow homomorphism, then there is a residual invariant subset $Y_0 \subset Y$ such that for any $y_0 \in Y_0$, $y \in Y$, and $\{t_n\} \subset \mathbb{R}$, if $y \cdot t_n \to y_0$, then for any $x_0 \in P^{-1}(y_0)$ there is $\{x_n\} \subset P^{-1}(y)$ such that $x_n \cdot t_n \to x_0$.*

Proof. See [28] or [29]. ∎

Remark 2.1. By Lemma 2.2, if $(Y, \mathbb{R})$ is minimal, P: $X \to Y$ is a flow homomorphism and there is a $y \in Y$ such that $P^{-1}(y)$ is a singleton, then there is a residual invariant subset $Y_0 \subset Y$ such that for any $y_0 \in Y_0$, $P^{-1}(y_0)$ is a singleton.

The following facts are immediate consequences from Definitions 2.2 and 2.3.

LEMMA 2.3. (1) *If $(X, \mathbb{R})$ is almost-periodic minimal, then any $x \in X$ is an almost-periodic point.*

(2) *If X is a 1-cover of Y and $(Y, \mathbb{R})$ is almost-periodic minimal, then so is $(X, \mathbb{R})$.*

Remark 2.2. If X is an almost 1-cover of Y with the flow homomorphism P: $X \to Y$ (i.e., there is $y \in Y$ such that $P^{-1}(y)$ is a singleton) and $(Y, \mathbb{R})$ is almost-periodic minimal, then for any $y_0 \in Y$ with $P^{-1}(y_0)$ being a singleton $\{x_0\}$, $x_0 \cdot t$ is almost automorphic in the sense that for any net α' there is a subnet α such that

$$T_{-\alpha} T_{\alpha} x_0 \cdot t = x_0 \cdot t$$

pointwise in $t \in \mathbb{R}$.

LEMMA 2.4. *Let $f \in C(\mathbb{R} \times D, \mathbb{R}^m)$, $(t, x) \mapsto f(t, x)$ $(D \subset \mathbb{R}^n)$, be uniformly almost periodic in t. Then*

(1) *$H(f)$ is compact in the compact open topology and is metrizable.*

(2) *$(H(f), \mathbb{R})$, $(g, t) \mapsto g \cdot t$ for $g \in H(f)$, is an almost-periodic minimal flow.*

Proof. See [21]. ∎

3. BASIC PROPERTIES

In this section, we collect some basic properties of solutions of (1.1) and (1.2) in terms of the associated skew-product flows. Let X and Π_t: $X \times X \times H(k, f) \to X \times X \times H(k, f)$ be as in (1.7) and π_t: $\mathbb{R} \times \mathbb{R} \times H(f) \to \mathbb{R} \times \mathbb{R} \times H(f)$ be as in (1.10). Denote by $\|\cdot\|$ either the norm in $X \times X$ or the norm in $\mathbb{R} \times \mathbb{R}$. We say $\Pi_t(u_0, v_0, l, g)(\pi_t(u_0, v_0, g))$ or $(u(t, \cdot\,; u_0, v_0, l, g), v(t, \cdot\,; u_0, v_0, l, g)$ $((u(t; u_0, v_0, g), v(t; u_0, v_0, g)))$ is *harmonically almost periodic* if $(u(t, \cdot\,; u_0, v_0, l, g), v(t, \cdot\,; u_0, v_0, l, g))$ $((u(t; u_0, v_0, g), v(t; u_0, v_0, g)))$ is uniformly almost periodic in t and $\mathscr{M}(u, v) \subset \mathscr{M}(k, f)(\mathscr{M}(u, v) \subset \mathscr{M}(f))$.

First of all, we define the following orderings on $X^+ \times X^+(\mathbb{R}^+ \times \mathbb{R}^+)$: for any $(u_1, v_1), (u_2, v_2) \in X^+ \times X^+$ $(\mathbb{R}^+ \times \mathbb{R}^+)$,

$$(u_1, v_1) \le_1 (u_2, v_2) \quad \text{if } u_1 \le u_2,\ v_1 \le v_2, \tag{3.1$_1$}$$

$$(u_1, v_1) <_1 (u_2, v_2) \quad \text{if } (u_1, v_1) \le_1 (u_1, v_2) \text{ and } (u_1, v_1) \ne (u_2, v_2), \tag{3.1$_2$}$$

$$(u_1, v_1) \ll_1 (u_2, v_2) \quad \text{if } (u_2 - u_1, v_2 - v_1) \in \operatorname{Int}(X^+) \times \operatorname{Int}(X^+) \quad (\operatorname{Int}\mathbb{R}^+ \times \operatorname{Int}\mathbb{R}^+), \tag{3.1$_3$}$$

$$(u_1, v_1) \le_2 (u_2, v_2) \quad \text{if } u_1 \le u_2,\ v_1 \ge v_2, \tag{3.2$_1$}$$

$$(u_1, v_1) <_2 (u_2, v_2) \quad \text{if } (u_1, v_1) \le_2 (u_2, v_2) \text{ and } (u_1, v_1) \ne (u_2, v_2), \tag{3.2$_2$}$$

$$(u_1, v_1) \ll_2 (u_2, v_2) \quad \text{if } (u_2 - u_1, v_1 - v_2) \in \operatorname{Int}(X^+) \times \operatorname{Int}(X^+) \quad (\operatorname{Int}\mathbb{R}^+ \times \operatorname{Int}\mathbb{R}^+), \tag{3.2$_3$}$$

where "$\le$" represents the standard order relation. For (u_1, v_1, l, g), $(u_2, v_2, l, g) \in X^+ \times X^+ \times H(k, f)$, we define

$$(u_1, v_1, l, g) \le_i (<_i, \ll_i)\, (u_2, v_2, l, g) \quad \text{if } (u_1, v_1) \le_i (<_i, \ll_i)\, (u_2, v_2) \ (i = 1, 2), \tag{3.3$_1$}$$

and

$$\|(u_1, v_1, l, g) - (u_2, v_2, l, g)\| = \|(u_1, v_1) - (u_2, v_2)\|. \tag{3.3$_2$}$$

Similarly, for (u_1, v_1, g), $(u_2, v_2, g) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, we define

$$(u_1, v_1, g) \le_i (<_i, \ll_i)(u_2, v_2, g) \quad \text{if } (u_1, v_1) \le_i (<_i, \ll_i)(u_2, v_2) \quad (i = 1, 2), \tag{3.4$_1$}$$

and

$$\|(u_1, v_1, g) - (u_2, v_2, g)\| = \|(u_1, v_1) - (u_2, v_2)\|. \tag{3.4$_2$}$$

Remark 3.1. For any (u_1, v_1), $(u_2, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, (u_1, v_1) and (u_2, v_2) are ordered with respect to $\leq_1$ or $\leq_2$, that is, $(u_1, v_1) \leq_1 (u_2, v_2)$ or $(u_2, v_2) \leq_1 (u_1, v_1)$ or $(u_1, v_1) \leq_2 (u_2, v_2)$ or $(u_2, v_2) \leq_2 (u_1, v_1)$.

Next, we explore some basic properties of solutions of (1.1). Clearly, the standard theory for parabolic equations yields

LEMMA 3.1.

(1) $\Pi_t(\mathbb{R}^+ \times \mathbb{R}^+ \times H(k, f)) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times H(k, f)$ *for* $t > 0$.

(2) $\Pi_t(X^+ \times \{0\} \times H(k, f)) \subset X^+ \times \{0\} \times H(k, f)$ *for* $t > 0$.

(3) $\Pi_t(X^+ \times X^+ \times H(k, f)) \subset X^+ \times X^+ \times H(k, f)$ *for* $t > 0$.

By Lemma 3.1, $(3.2)_{1-3}$, and the comparison principle for parabolic equations, we have

LEMMA 3.2. *If* $(u_1, v_1), (u_2, v_2) \in X^+ \times X^+$ *and* $(u_1, v_1) \leq_2 (u_2, v_2)$, *then* $\Pi_t(u_1, v_1, l, g) \leq_2 \Pi_t(u_2, v_2, l, g)$ *for any* $t > 0$ *and* $(l, g) \in H(k, f)$. *Moreover, if* $(u_1, v_1) <_2 (u_2, v_2)$ *and* $(u_1, v_1) \notin X^+ \times \{0\}$, $(u_2, v_2) \notin \{0\} \times X^+$, *then* $\Pi_t(u_1, v_1, l, g) \ll_2 \Pi_t(u_2, v_2, l, g)$ *for any* $t > 0$ *and* $(l, g) \in H(k, f)$.

Note that by Lemma 3.1, $X^+ \times \{0\} \times H(k, f)$ and $\{0\} \times X^+ \times H(k, f)$ are invariant under Π_t. Clearly, on $X^+ \times \{0\} \times H(k, f)$ and $\{0\} \times X^+ \times H(k, f)$, Π_t is generated by

$$\begin{aligned} u_t &= k_1(t, x)\Delta u + u f_1(t, u, 0), \qquad && x \in \Omega \\ \frac{\partial u}{\partial n} &= 0, && x \in \partial\Omega \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} v_t &= k_2(t, x)\Delta v + v f_2(t, 0, v), \qquad && x \in \Omega \\ \frac{\partial v}{\partial n} &= 0, && x \in \partial\Omega, \end{aligned} \tag{3.6}$$

respectively. Following from [22], we have

LEMMA 3.3. (1) *There is* $E_1 (\subset \mathrm{Int}(\mathbb{R}^+) \times \{0\} \times H(k, f))$, *which is invariant under* Π_t, *and a* 1-*cover of* $H(k, f)$ *with respect to* P, *hence of form*

$$E_1 = \{(u_g, 0, l, g) : (l, g) \in H(k, f)\}, \tag{3.7}$$

and attracting in the sense that, for any $u_0 \in X^+$ $(u_0 \neq 0)$ *and* $(l, g) \in H(k, f)$,

$$\|u(t,\cdot\,; u_0, 0, l, g) - u(t; u_g, 0, g)\| \to 0$$

as $t \to \infty$.

(2) *There is* $E_2(\subset \{0\} \times Int(\mathbb{R}^+) \times H(k, f))$, *which is invariant under* Π_t, *and a* 1-*cover of* $H(k, f)$ *with respect to* P, *hence, of form*

$$E_2 = \{(0, v_g, l, g) : (l, g) \in H(k, f)\}, \tag{3.8}$$

and attracting in the sense that, for any $v_0 \in X^+$ $(v_0 \neq 0)$ *and* $(l, g) \in H(k, f)$,

$$\|v(t,\cdot\,; 0, v_0, l, g) - v(t; 0, v_g, g)\| \to 0$$

as $t \to \infty$.

Remark 3.2. By Lemmas 2.1 and 2.3, $u(t; u_g, 0, g)$ and $v(t; 0, v_g, g)$ are harmonically almost periodic.

Let $E \subset X^+ \times X^+ \times H(k, f)$ be such that

$$E \cap P^{-1}(l, g) = \left([0, u_g] \times [0, v_g] \times \{l, g\}\right) \setminus \{(0, 0, l, g)\}, \tag{3.9}$$

where $[0, u_g] = \{u \in X \mid 0 \le u \le u_g\}$ and $[0, v_g] = \{v \in X \mid 0 \le v \le v_g\}$. Then by Lemmas 3.2 and 3.3 and (1.5), we have

LEMMA 3.4. (1) $\Pi_t E \subset E$ *for any* $t > 0$.

(2) *For any* $(u_0, v_0) \in X^+ \times X^+$ $((u_0, v_0) \neq (0, 0))$ *and* $(l, g) \in H(k, f)$, $\omega(u_0, v_0, l, g) \subset E$.

Now, we explore the basic properties of (1.2). Note that every solution of $(1.9)_g$ can be extended backward. Given $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $g \in H(f)$, define $I(u_0, v_0, g)$ as

$$I(u_0, v_0, g) = \text{existence interval of } \left(u(t; u_0, u_0, g), v(t; u_0, v_0, g)\right). \tag{3.10}$$

In order to consider the behavior of solutions of $(1.9)_g$ in the backward direction, let $\xi(t) = u(-t)$ and $\eta(t) = v(-t)$. Then $(\xi(t), \eta(t))$ satisfies

$$\begin{aligned} \xi_t &= \xi \tilde{g}_1(t, \xi, \eta) \\ \eta_t &= \eta \tilde{g}_2(t, \xi, \eta), \end{aligned} \tag*{$(3.11)_g$}$$

where $\tilde{g}_i(t, \xi, \eta) = -g_i(-t, \xi, \eta)$ $(i = 1, 2)$. By (1.4), $(3.11)_g$ is a cooperative system for $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+$. Hence we have

LEMMA 3.5. (1) *For any* $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ *and any* $g \in H(f)$, $(u(t; u_0, v_0, g), v(t; u_0, v_0, g)) \in \mathbb{R}^+ \times \mathbb{R}^+$ *for* $t \in I(u_0, v_0, g)$.

(2) *For any* (u_1, v_1), $(u_2, v_2) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+)$ *with* $(u_1, v_1) \le_1 (<_1)(u_2, v_2)$,

$$(u(t; u_1, v_1, g), v(t; u_1, v_1, g)) \\ \le_1 (\ll_1)(u(t; u_2, v_2, g), v(t; u_2, v_2, g)) \tag{3.12}$$

for $t \in I(u_1, v_1, g) \cap I(u_2, v_2, g)$ *with* $t < 0$.

Now, given $(u_0, v_0) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+)$ and $g \in H(f)$, let

$$P_1(u_0, v_0, g) = \{(u, v, g) \mid (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+, (u_0, v_0) \ll_1 (u, v)\}, \tag*{$(3.13)_1$}$$

$$P_2(u_0, v_0, g) = \{(u, v, g) \mid (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+, (u, v) \ll_2 (u_0, v_0)\}, \tag*{$(3.13)_2$}$$

$$P_3(u_0, v_0, g) = \{(u, v, g) \mid (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+, (u, v) \ll_1 (u_0, v_0)\}, \tag*{$(3.13)_3$}$$

$$P_4(u_0, v_0, g) = \{(u, v, g) \mid (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+, (u_0, v_0) \ll_2 (u, v)\}. \tag*{$(3.13)_4$}$$

Then, by Lemma 3.2, we have

LEMMA 3.6. (1) $\pi_t(\overline{P_i(u_0, v_0, g)} \setminus \{(u_0, v_0, g)\}) \subset P_i(\pi_t(u_0, v_0, g))$ *for* $i = 2, 4$ *and* $t > 0$.

(2) *For* $(u_1, v_1, g) \in P_k(u_0, v_0, g)$ *with* $k = 1$ (*respectively*, $k = 3$), *only one of the following alternatives is met*:

(i) *there is* $t' > 0$ *such that* $\pi_t(u_1, v_1, g) \in \overline{P_j(\pi_t(u_0, v_0, g))}$ *for* $j = 2$ *or* 4 *and* $t \ge t'$;

(ii) $\pi_t(u_1, v_1, g) \in P_k(\pi_t(u_0, v_0, g))$ *for* $t > 0$.

By Lemma 3.5, we have

LEMMA 3.7. (1) *For any* $(u_1, v_1, g) \in \overline{P_i(u_0, v_0, g)} \setminus \{(u_0, v_0, g)\}$, $\pi_t(u_1, v_1, g) \in P_i(\pi_t(u_0, v_0, g))$ *for* $i = 1, 3$ *and* $t \in I(u_0, v_0, g) \cap I(u_1, v_1, g)$ *with* $t < 0$.

(2) *For* $(u_1, v_1, g) \in P_k(u_0, v_0, g)$ *with* $k = 2$ (*respectively*, $k = 4$), *only one of the following alternatives is met*:

(i) *there is* $t' < 0$ *such that* $\pi_t(u_1, v_1, g) \in \overline{P_j(\pi_t(u_0, v_0, g))}$ *for* $j = 1$ *or* 3 *and* $t \in I(u_0, v_0, g) \cap I(u_1, v_1, g)$ *with* $t \leq t'$.

(ii) $\pi_t(u_1, v_1, g) \in P_k(\pi_t(u_0, v_0, g))$ *for* $t \in I(u_0, v_0, g) \cap I(u_1, v_1, g)$ *with* $t < 0$.

By Lemmas 3.6 and 3.7, we have

LEMMA 3.8. (1) *Given* $(u_0, v_0, g) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+) \times H(f)$, *if* $\|\pi_t(u_0, v_0, g) - \pi_t(u_g, 0, g)\| \to 0$ $(\|\pi_t(u_0, v_0, g) - \pi_t(0, v_g, g)\| \to 0)$ *as* $t \to \infty$, *then*

$$\|\pi_t(u_1, v_1, g) - \pi_t(u_g, 0, g)\| \to 0 \qquad \big(\|\pi_t(u_1, v_1, g) - \pi_t(0, v_g, g)\| \to 0\big)$$

as $t \to \infty$ *for any* $(u_1, v_1, g) \in \overline{P_4(u_0, v_0, g)}(\overline{P_2(u_0, v_0, g)})$.

(2) *Given* $(u_0, v_0, g) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+) \times H(f)$, *if* $\|\pi_t(u_0, v_0, g) - (0, 0, g \cdot t)\| \to 0$ *as* $t \to -\infty$, *then*

$$\|\pi_t(u_1, v_1, g) - (0, 0, g \cdot t)\| \to 0$$

as $t \to -\infty$ *for any* $(u_1, v_1, g) \in \overline{P_3(u_0, v_0, g)}$. *If* $\|\pi_t(u_0, v_0, g)\| \to \infty$ *as* $t \to \inf\{t \mid t \in I(u_0, v_0, g)\}$, *then*

$$\|\pi_t(u_1, v_1, g)\| \to \infty$$

as $t \to \inf\{t \mid t \in I(u_1, v_1, g)\}$ *for any* $(u_1, v_1, g) \in \overline{P_1(u_0, v_0, g)}$.

In the remaining sections of the paper, we shall only study positive solutions of (1.1) ((1.2)) or positive motions of (1.7) ((1.10)), that is, $\Pi_t(u_0, v_0, l, g)$ $(\pi_t(u_0, v_0, g))$ with $(u_0, v_0) \in (X^+ \times X^+) \setminus \{(0,0)\}$ and $(l, g) \in H(k, f)$ $((u_0, v_0) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \{(0,0)\}$ and $g \in H(f))$.

4. ASYMPTOTIC BEHAVIOR OF SPATIALLY HOMOGENEOUS SOLUTIONS

In this section, we shall study the asymptotic behavior of positive spatially homogeneous solutions of (1.1), i.e., positive solutions of (1.2). As mentioned in the Introduction, this is equivalent to investigating the structure of the ω-limit sets of positive motions of π_t.

By Lemma 3.5, for any positive motion $\pi_t(u_0, v_0, g)$ the ω-limit set $\omega(u_0, v_0, g)$ is *positive* in the sense that $\omega(u_0, v_0, g) \subset (\mathbb{R}^+ \times \mathbb{R}^+ \times H(f)) \setminus (\{0\} \times \{0\} \times H(f))$. In general, given $K \subset \mathbb{R} \times \mathbb{R} \times H(f)$, we say K is *positive* (*strictly positive*) if $K \subset (\mathbb{R}^+ \times \mathbb{R}^+ \times H(f)) \setminus (\{0\} \times \{0\} \times H(f))$

$(\mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+) \times H(f))$. A positive set K is said to be *totally ordered* with respect to $\le_1$ $(\le_2)$ if for any (u_1, v_1, g) and $(u_2, v_2, g) \in K$, $(u_1, v_1, g) \le_1 (\le_2)(u_2, v_2, g)$ or $(u_2, v_2, g) \le_1 (\le_2)(u_2, v_2, g)$. Given $K_1, K_2 \subset \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, we say K_1 and K_2 are *ordered* with respect to $\le_1$ $(\le_2)$ if $(u_1, v_1, g) \le_1 (\le_2)(u_2, v_2, g)$ or $(u_2, v_2, g) \le_1 (\le_2)(u_1, v_1, g)$ for any $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$. Two positive solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ of $(1.9)_g$ are said to be *ordered with respect to* $\le_1$ $(\le_2)$ if $(u_1(t), v_1(t)) \le_1 (\le_2)(u_2(t), v_2(t))$ or $(u_2(t), v_2(t)) \le_1 (\le_2)(u_1(t), v_1(t))$ for $t \ge 0$.

The following theorem describes the main results of this section.

THEOREM 4.1. (1) *Any positive minimal set of π_t is an almost* 1-*cover of $H(f)$ and is ordered with respect to $\le_2$.*

(2) *Any two positive minimal sets of π_t are ordered with respect to $\le_2$.*

(3) *For any $(u_0, v_0, g_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, $\omega(u_0, v_0, g)$ contains at most two minimal sets and is ordered with respect to $\le_2$.*

(4) *Any positive almost periodic solution $(u(t), v(t))$ of* (1.2) (*if it exists*) *is harmonic.*

Remark 4.1. If K is a positive minimal set of π_t, then for any $g \in H(f)$ with $K \cap P^{-1}(g) = \{(u_0, v_0, g)\}$ being a singleton, $(u(t; u_0, v_0, g), v(t; u_0, v_0, g))$ is an almost automorphic solution of $(1.9)_g$. If K is indeed almost periodic minimal, then for any $g \in H(f)$, $K \cap P^{-1}(g) = \{(u_0, v_0, g)\}$ is a singleton and $(u(t; u_0, v_0, g), v(t; u_0, v_0, g))$ is a harmonically almost-periodic solution of $(1.9)_g$.

Before proving the theorem we establish

LEMMA 4.2. (1) *Suppose that (u_1, v_1, g), $(u_2, v_2, g) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+) \times H(f)$ are such that $\pi_t(u_1, v_1, g) <_1 \pi_t(u_2, v_2, g)$ for $t > 0$. Then*

$$\ln u_1(t) - \ln u_2(t) > \ln u_1(s) - \ln u_2(s)$$

and

$$\ln v_1(t) - \ln v_2(t) > \ln v_1(s) - \ln v_2(s)$$

for any $0 < s < t$, where $(u_i(t), v_i(t)) = (u(t; u_i, v_i, g), v(t; u_i, v_i, g))$ $(i = 1, 2)$. Moreover, if $\pi_t(u_1, v_1, g)$ and $\pi_t(u_2, v_2, g)$ are bounded away from $\mathbb{R}^+ \times \{0\} \times H(f)$ and $\{0\} \times \mathbb{R}^+ \times H(f)$, respectively, that is, there is $\delta > 0$ such that

$$(\delta, \delta, g) \le_1 \pi_t(u_i, v_i, g) \ (i = 1, 2) \qquad \text{for } t > 0, \tag{4.1}$$

then

$$\|(u_1(t), v_1(t)) - (u_2(t), v_2(t))\| \to 0 \qquad \text{as } t \to \infty.$$

(2) *For any* $(u_1, v_1, g), (u_2, v_2, g) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+) \times H(f)$ *with* $(u_1, v_1, g) <_1 (u_2, v_2, g)$,

$$\ln u_1(t) - \ln u_2(t) < \ln u_1(s) - \ln u_2(s)$$

and

$$\ln v_1(t) - \ln v_2(t) < \ln v_1(s) - \ln v_2(s)$$

for any $t < s < 0$, $t, s \in I(u_1, v_1, g) \cap I(u_2, v_2, g)$. *Moreover, if* $\pi_t(u_1, v_1, g)$ *or* $\pi_t(u_2, v_2, g)$ *is bounded away from* $\{0\} \times \{0\} \times H(f)$ *for* $t \in I(u_1, v_1, g)$ *or* $I(u_2, v_2, g)$, *then*

$$\lim_{t \to t_0} \|(u_1(t), v_1(t)) - (u_2(t), v_2(t))\| > 0,$$

where $t_0 = inf(I(u_1, v_1, g) \cap I(u_2, v_2, g))$.

Proof. (1) Note that

$$(\ln u_i(t))_t = f_1(t, u_i(t), v_i(t))$$

and

$$(\ln v_i(t))_t = f_2(t, u_i(t), v_i(t))$$

for $i = 1, 2$. Then, by (1.4) and the assumption $\pi_t(u_1, v_1, g) <_1 \pi_t(u_2, v_2, g)$ for $t > 0$,

$$\left(\ln \frac{u_1(t)}{u_2(t)}\right)_t = f_1(t, u_1(t), v_1(t)) - f_1(t, u_2(t), v_2(t)) > 0 \quad (4.2)$$

and

$$\left(\ln \frac{v_1(t)}{v_2(t)}\right)_t = f_2(t, u_1(t), v_1(t)) - f_2(t, u_2(t), v_2(t)) > 0$$

for $t > 0$. Therefore,

$$\ln \frac{u_1(t)}{u_2(t)} > \ln \frac{u_1(s)}{u_2(s)} \text{ and } \ln \frac{v_1(t)}{v_2(t)} > \ln \frac{v_1(s)}{v_2(s)} \qquad \text{for } t > s > 0.$$

That is,

$$\ln u_1(t) - \ln u_2(t) > \ln u_1(s) - \ln u_2(s) \qquad \text{and}$$
$$\ln v_1(t) - \ln v_2(t) > \ln v_1(s) - \ln v_2(s)$$

for $t > s > 0$.

Now assume (4.1). We claim that

$$\lim_{t\to\infty} \ln \frac{u_1(t)}{u_2(t)} = \lim_{t\to\infty} \ln \frac{v_1(t)}{v_2(t)} = 0.$$

For otherwise, if $\lim_{t\to\infty} \ln \frac{u_1(t)}{u_2(t)} = r < 0$, then by (4.1) there is $\tilde{\delta} > 0$ such that $u_1(t) - u_2(t) \le -\tilde{\delta}$ for $t \gg 1$. This together with (1.4) and (4.2) implies that $\lim_{t\to\infty} \ln \frac{u_1(t)}{u_2(t)} = \infty$, a contradiction. Hence, $\lim_{t\to\infty} \ln(u_1(t)/u_2(t)) = 0$. Similarly, $\lim_{t\to\infty} \ln(v_1(t)/v_2(t)) = 0$. Therefore,

$$\lim_{t\to\infty} \frac{u_1(t)}{u_2(t)} = \lim_{t\to\infty} \frac{v_1(t)}{v_2(t)} = 1.$$

It then follows that $\|\Pi_t(u_1, u_2, g) - \Pi_t(u_2, v_2, g)\| \to 0$ as $t \to \infty$.

(2) First, by Lemma 3.5, $\pi_t(u_1, v_1, g) \ll_1 \pi_t(u_2, v_2, g)$ for $t < 0$, $t \in I(u_1, v_1, g) \cap I(u_2, v_2, g)$. Then, by arguments similar to those above, we have

$$\ln \frac{u_1(t)}{u_2(t)} < \ln \frac{u_1(s)}{u_2(s)} \qquad \text{and} \qquad \ln \frac{v_1(t)}{v_2(t)} < \ln \frac{v_1(s)}{v_2(s)},$$

that is,

$$\ln u_1(t) - \ln u_2(t) < \ln u_1(s) - \ln u_2(s) \qquad \text{and}$$
$$\ln v_1(t) - \ln v_2(t) < \ln v_1(s) - \ln v_2(s)$$

for $t < s < 0$, $t, s \in I(u_1, v_1, g) \cap I(u_2, v_2, g)$.

Now let $t_0 = \inf(I(u_1, v_1, g) \cap I(u_2, v_2, g))$. Then, by the above arguments,

$$\lim_{t\to t_0} \ln \frac{u_1(t)}{u_2(t)} = r_1 < 0 \qquad \text{and} \qquad \lim_{t\to t_0} \ln \frac{v_1(t)}{v_2(t)} = r_2 < 0. \tag{4.3}$$

Without loss of generality, we assume that $\pi_t(u_2, v_2, g)$ is bounded away from $\{0\} \times \{0\} \times H(f)$. Note that

$$\begin{aligned} &|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)| \\ &\quad = \left| \frac{u_1(t)}{u_2(t)} - 1 \right| u_2(t) + \left| \frac{v_1(t)}{v_2(t)} - 1 \right| v_2(t). \end{aligned}$$

This together with (4.3) implies that

$$\liminf_{t\to t_0} \|\pi_t(u_1, v_1, g) - \pi_t(u_2, v_2, g)\| > 0.$$

∎

Proof of Theorem 4.1. (1) Suppose that $K \subset \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$ is a positive minimal set of π_t.

First, we prove that K is an almost 1-cover of $H(f)$. If $K \cap (\mathbb{R}^+ \times \{0\} \times H(f)) \neq \emptyset (K \cap (\{0\} \times R^+ \times H(f)) \neq \emptyset)$, then by the minimality of K and Lemma 3.3, $K \subset \mathbb{R}^+ \times \{0\} \times H(f)$ $(K \subset \{0\} \times \mathbb{R}^+ \times H(f))$ and hence K is a 1-cover of $H(f)$.

We then assume that $K \subset \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+) \times H(f)$. Let $H_0(f) \equiv Y_0 \subset H(f)$ be as in Lemma 2.2 with $X = K$, $Y = H(f)$, and P being as in (1.12). We prove that $K \cap P^{-1}(g)$ is a singleton for $g \in H_0(f)$ (hence K is an almost 1-cover of $H(f)$). By Remark 3.1, for any (u_1, v_1, g), $(u_2, v_2, g) \in K$, (u_1, v_1) and (u_2, v_2) are ordered with respect to $\leq_1$ or $\leq_2$. It then suffices to prove that $K \cap P^{-1}(g)$ contains no ordered pair with respect to $\leq_1$ or $\leq_2$ for $g \in H_0(f)$.

Suppose that $K \cap P^{-1}(g_0)$ has an ordered pair with respect to $\leq_2$ for some $g_0 \in H_0(f)$. Without loss of generality, we may assume that there are $(u_1, v_1, g_0), (u_2, v_2, g_0) \in K$ such that $(u_1, v_1, g_0) \ll_2 (u_2, v_2, g_0)$. Given $(u_0, v_0, g_0) \in K$, let $t_n \to -\infty$ be such that $\pi_{t_n}(u_0, v_0, g_0) \to (u_1, v_1, g_0)$. By Lemma 2.2, there are $(u^n, v^n, g_0) \in K$ such that $\pi_{t_n}(u^n, v^n, g_0) \to (u_2, v_2, g_0)$. Then $\pi_{t_n}(u_0, v_0, g_0) \ll_2 \pi_{t_n}(u^n, v^n, g_0)$ for $n \gg 1$. This together with Lemma 3.2 implies that $(u_0, v_0, g_0) \ll_2 (u^n, v^n, g_0)$ for $n \gg 1$. Therefore, the set

$$A(u_0, v_0, g_0) = \{(u, v, g_0) \in K : (u_0, v_0, g_0) \ll_2 (u, v, g_0)\}$$

is not empty. Hence, for any $(u_0, v_0, g_0) \in K$, there are (u^*, v^*, g_0) and a neighborhood $U(u_0, v_0, g_0)$ of (u_0, v_0, g_0) in $K \cap P^{-1}(g_0)$ such that $(u, v, g_0) \ll_2 (u^*, v^*, g_0)$ for any $(u, v, g_0) \in U(u_0, v_0, g_0)$. By the finite open covering property, there are (u_i, v_i, g_0) $(i = 1, 2, 3, \dots, m)$ and (u_i^*, v_i^*, g_0) such that $\bigcup_{i=1}^m U(u_i, v_i, g_0) = K \cap P^{-1}(g_0)$ and $(u, v, g_0) \ll_2 (u_i^*, v_i^*, g_0)$ for any $(u, v, g_0) \in U(u_i, v_i, g_0)$. Let (u^*, v^*, g_0) be the maximal element of $\{(u_i^*, v_i^*, g_0)\}_{i=1,2,\dots,m}$ with respect to $\leq_2$. Then (u^*, v^*, g_0) is the maximal element of $K \cap P^{-1}(g_0)$. Now let $s_n \to -\infty$ be such that $\pi_{s_n}(u^*, v^*, g_0) \to (u_1, v_1, g_0)$. By Lemma 2.2 again, there are $(u_n^*, v_n^*, g_0) \in K \cap P^{-1}(g_0)$ such that $\pi_{s_n}(u_n^*, v_n^*, g_0) \to (u_2, v_2, g_0)$. Then by arguments similar to these above, $(u^*, v^*, g_0) \ll_2 (u_n^*, v_n^*, g_0)$ for n large enough, which is contradictory to the fact that (u^*, v^*, g_0) is the maximal element of $K \cap P^{-1}(g_0)$. Hence, there is no $\leq_2$ ordered pair on $K \cap P^{-1}(g)$ for any $g \in H_0(f)$.

Similarly, we can prove that there is no $\leq_1$ ordered pair on $K \cap P^{-1}(g)$ for any $g \in H_0(f)$.

Therefore, $K \cap P^{-1}(g_0)$ is a singleton for any $g \in H_0(f)$ and K is an almost 1-cover of $H(f)$.

Next, we prove that K is ordered with respect to $\leq_2$. If $K \subset \mathbb{R}^+ \times \{0\} \times H(f)$ or $\{0\} \times \mathbb{R}^+ \times H(f)$, then K is a 1-cover of $H(f)$ and hence is ordered with respect to $\leq_2$. We then assume that $K \subset \operatorname{Int}(\mathbb{R}^+) \times \operatorname{Int}(\mathbb{R}^+) \times H(f)$. By Remark 3.1, again, it is sufficient to prove that for any $g \in H(f)$, $K \cap P^{-1}(g)$ has no $\leq_1$ ordered pair. Suppose that there are $(u_1, v_1, g), (u_2, v_2, g) \in K$ with $(u_1, v_1, g) <_1 (u_2, v_2, g)$. Clearly, both $\pi_t(u_1, v_1, g)$ and $\pi_t(u_2, v_2, g)$ are bounded away from $\{0\} \times \{0\} \times H(f)$. Hence, by Lemma 4.2(2),

$$\liminf_{t \to -\infty} \|\pi_t(u_1, v_1, g) - \pi_t(u_2, v_2, t)\| > 0.$$

Since K is an almost 1-cover of $H(f)$, there is $t_n \to -\infty$ such that $\|\pi_{t_n}(u_1, v_1, g) - \pi_{t_n}(u_2, v_2, g)\| \to 0$ as $n \to \infty$, a contradiction. Therefore, K is ordered with respect to $\leq_2$.

(2) Suppose that K_1 and K_2 are two positive minimal sets of π_t. Without loss of generality, we assume that both K_1 and K_2 are bounded away from $\mathbb{R}^+ \times \{0\} \times H(f)$ and $\{0\} \times \mathbb{R}^+ \times H(f)$. We prove that either $(u_1, v_1, g) <_2 (u_2, v_2, g)$ or $(u_2, v_2, g) <_2 (u_1, v_1, g)$ for any $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$.

First, we claim that, for any $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$, $(u_1, v_1, g) <_2 (u_2, v_2, g)$ or $(u_2, v_2, g) <_2 (u_1, v_1, g)$. Assume that there are $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$ such that $(u_1, v_1, g) <_1 (u_2, v_2, g)$. Then $\pi_t(u_1, v_1, g) <_1 \pi_t(u_2, v_2, g)$ for $t < 0$. Let $t_n \to -\infty$ be such that $\pi_{t_n}(u_1, v_1, g) \to (u_1^*, v_1^*, g^*) \in K_1$ and $\pi_{t_n}(u_2, v_2, g) \to (u_2^*, v_2^*, g^*) \in K_2$. Then $(u_1^*, v_1^*, g^*) <_1 (u_2^*, v_2^*, g^*)$ and

$$\begin{aligned}\pi_t(u_1^*, v_1^*, g^*) &= \lim_{n \to \infty} \pi_{t+t_n}(u_1, v_1, g) <_1 \lim_{n \to \infty} \pi_{t+t_n}(u_2, v_2, g) \\ &= \pi_t(u_2^*, v_2^*, g^*)\end{aligned}$$

for $t \geq 0$. By Lemma 4.2(1), we have $\|\pi_t(u_1^*, v_1^*, g^*) - \pi_t(u_2^*, v_2^*, g^*)\| \to 0$ as $t \to \infty$, a contradiction. Therefore for any $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$, we have $(u_1, v_2, g) <_2 (u_2, v_2, g)$ or $(u_2, v_2, g) <_2 (u_1, v_1, g)$.

Now we claim that if $(u_1^*, v_1^*, g^*) <_2 (u_2^*, v_2^*, g^*)$ ($(u_2^*, v_2^*, g^*) <_2 (u_1^*, v_1^*, g^*)$) for some $(u_1^*, v_1^*, g^*) \in K_1$ and $(u_2^*, v_2^*, g^*) \in K_2$, then $(u_1, v_1, g) <_2 (u_2, v_2, g)$ ($(u_2, v_2, g) <_2 (u_1, v_1, g)$) for any $(u_1, v_1, g) \in K_1$ and $(u_2, v_2, g) \in K_2$. For otherwise there are $(u_i^*, v_i^*, g^*) \in K_i$ and $(u_i^{**}, v_i^{**}, g^{**}) \in K_i$ ($i = 1, 2$) such that $(u_1^*, v_1^*, g^*) <_2 (u_2^*, v_2^*, g^*)$ and $(u_2^{**}, v_2^{**}, g^{**}) <_2 (u_1^{**}, v_1^{**}, g^{**})$. By (1), there is $g_0 \in H(f)$ such that $(u_i, v_i, g_0) = K_i \cap P^{-1}(g_0)$ ($i = 1, 2$). By the minimality, there is $t_n^* \to \infty$ such that $\pi_{t_n^*}(u_i^*, v_i^*, g^*) \to (u_i, v_i, g_0)$ ($i = 1, 2$) as $n \to \infty$. Hence, $(u_1, v_1, g_0) <_2 (u_2, v_2, g_0)$. Similarly, there is $t_n^{**} \to \infty$ such that $\pi_{t_n^{**}}(u_i^{**}, v_i^{**}, g^{**}) \to (u_i, v_i, g_0)$ as $n \to \infty$. Then $(u_2, v_2, g_0) <_2 (u_1, v_1, g_0)$, a contradiction. Part 2 thus follows.

(3) Without loss of generality, assume that $(u_0, v_0) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+)$. By Lemma 3.4, $\omega(u_0, v_0, g_0)$ is positive. Suppose $\omega(u_0, v_0, g_0)$ contains three distinct minimal sets K_1, K_2, K_3. By Part 2, we may assume that there are $(u_i, v_i, g_0) \in K$ $(i = 1, 2, 3)$ such that $(u_1, v_1, g_0) \ll_2 (u_2, v_2, g_0) \ll_2 (u_3, v_3, g_0)$. Hence there is $T > 0$ such that $\pi_T(u_0, v_0, g_0) \in P_2(\pi_T(u_2, v_2, g_0))$. This together with Lemma 3.2 implies that there is no $t_n \to \infty$ such that $\pi_{t_n}(u_0, v_0, g_0) \to (u_3, v_3, g_0)$, a contradiction. Hence, $\omega(u_0, v_0, g_0)$ contains at most two minimal sets.

Now we prove that $\omega(u_0, v_0, g_0)$ is ordered with respect to $\leq_2$. Suppose that there are $(u_1, v_1, g), (u_1, v_2, g) \in \omega(u_0, v_0, g_0)$ such that $(u_1, v_1, g) <_1 (u_2, v_2, g)$. Clearly, both $\pi_t(u_1, v_1, g)$ and $\pi_t(u_2, v_2, g)$ are bounded away from $\{0\} \times \{0\} \times H(f)$. Then by Lemma 4.2(2),

$$\liminf_{t \to -\infty} \|\pi_t(u_1, v_1, g) - \pi_t(u_2, v_2, g)\| > 0. \tag{4.4}$$

We intend to show that this yields a contradiction.

First, suppose that $\omega(u_0, v_0, g_0)$ contains only one minimal set K. If $(u_1, v_1, g) \in K$, then by (4.4), $\alpha(u_2, v_2, g) \cap K = \emptyset$, where $\alpha(u_2, v_2, g)$ is the α-limit set of $\pi_t(u_2, v_2, g)$. This implies that $\omega(u_0, v_0, g_0)$ contains at least two minimal sets, a contradiction. If $(u_1, v_1, g), (u_2, v_2, g) \notin K$, then by Part 1 and (4.4), there is $t_n \to -\infty$ such that $(u_1^*, v_1^*, g^*) = \lim_{n \to \infty} \pi_{t_n}(u_1, v_1, g) \in K$ and $(u_1^*, v_1^*, g^*) <_1 (u_2^*, v_2^*, g^*) = \lim_{n \to \infty} \pi_{t_n}(u_2, v_2, g) \notin K$. By the above arguments, this again yields a contradiction. Therefore, if $\omega(u_0, v_0, g_0)$ contains only one minimal set, then $\omega(u_0, v_0, g_0)$ is ordered with respect to $\leq_2$.

Next, suppose that $\omega(u_0, v_0, g_0)$ contains two minimal sets K_1 and K_2. Then by Parts 1 and 2 above, if $(u_1, v_1, g) \in K_1 \cup K_2$, then $(u_2, v_2, g) \notin K_1 \cup K_2$. By (4.4) and Part 2, $\alpha(u_2, v_2, g) \cap (K_1 \cup K_2) = \emptyset$. This implies that $\omega(u_0, v_0, g_0)$ contains at least three minimal sets, a contradiction. If (u_1, v_1, g), $(u_2, v_2, g) \notin K_1 \cup K_2$, then by (4.4), and Parts 1 and 2, there is $t_n \to -\infty$ such that $(u_1^*, v_1^*, g^*) = \lim_{n \to \infty} \pi_{t_n}(u_1, v_1, g) \in K_1 \cup K_2$ and $(u_1^*, v_1^*, g^*) <_1 (u_2^*, v_2^*, g^*) = \lim_{n \to \infty} \pi_{t_n}(u_2, v_2, g) \notin K_1 \cup K_2$. This again yields a contradiction. Therefore, if $\omega(u_0, v_0, g_0)$ contains two minimal sets, then $\omega(u_0, v_0, g)$ is also ordered with respect to $\leq_2$. Part 3 thus follows.

(4) It follows from Part 1 and Lemma 2.1. ▮

COROLLARY 4.3. *If f is periodic of period T, then for any $(u_0, v_0, g_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$, $\omega(u_0, v_0, g_0)$ is periodic minimal with period T.*

Proof. It is sufficient to prove that $\omega(u_0, v_0, g_0)$ is a 1-cover of $H(f)$. Without loss of generality, assume $(u_0, v_0) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+)$. By Theorem 4.1(3), $\omega(u_0, v_0, g_0)$ is ordered with respect to $\leq_2$. Suppose that

$\omega(u_0, v_0, g_0)$ is not a 1-cover of $H(f)$. Then there are (u_1, v_1, g_0), $(u_2, v_2, g_0) \in \omega(u_0, v_0, g_0)$ such that $(u_1, v_1, g_0) <_2 (u_2, v_2, g_0)$. By Lemma 3.3, if $(u_1, v_1, g_0) \in \mathbb{R}^+ \times \{0\} \times H(f)$ $(\{0\} \times \mathbb{R}^+ \times H(f))$, then $(u_2, v_2, g_0) \notin \mathbb{R}^+ \times \{0\} \times H(f)$ $(\{0\} \times \mathbb{R}^+ \times H(f))$. Then, by Lemma 3.2, $\pi_t(u_1, v_1, g_0) \ll_2 \pi_t(u_2, v_2, g_0)$ for $t > 0$. Without loss of generality, we may assume that $(u_1, v_1, g_0) \ll_2 (u_2, v_2, g_0)$. Now there is $n_k \to \infty$ such that $\pi_{n_k T}(u_0, v_0, g_0) \to (u_2, v_2, g_0)$ and there is $n_{k'} \to \infty$ such that $\pi_{n_{k'} T}(u_0, v_0, g_0) \to (u_1, v_1, g_0)$. Hence, there are $M < N$ such that $\pi_{MT}(u_0, v_0, g_0) \ll_2 \pi_{NT}(u_0, v_0, g_0)$. This implies that $\pi_{(M+n(N-M))T}(u_0, v_0, g_0)$ is monotone with respect to $\leq_2$. It then follows that $\omega(u_0, v_0, g_0)$ is minimal and hence is a 1-cover of $H(f)$, a contradiction. ∎

The following corollary follows directly from Theorem 4.1.

COROLLARY 4.4. *Any two positive almost-periodic solutions of* $(1.9)_g$ $(g \in H(f))$ *are ordered with respect to* $\leq_2$, *and hence all the positive almost-periodic solutions of* $(1.9)_g$ *form a totally ordered set with respect to* $\leq_2$.

COROLLARY 4.5. *Suppose that* $f_1(t, u, v) = f_2(t, u, v) \equiv f^*(t, u + v)$. *Let* $\Gamma(g) = \{(u, v, g) : 0 \leq u,\ v \leq u_g,\ u + v = u_g\}$ $(g \in H(f))$. *Then*

(1) $u_g = v_g$ *for any* $g \in H(f)$, *and for any* $(u_0, v_0, g) \in \Gamma(g)$, $\pi_t(u_0, v_0, g)$ *is almost periodic.*

(2) *For any* $(u_0, v_0, g_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$ $((u_0, v_0) \neq (0, 0))$, $\omega(u_0, v_0, g_0)$ *is almost-periodic minimal and* $\omega(u_0, v_0, g_0) \cap P^{-1}(g) \subset \Gamma(g)$.

Proof. (1) Let $w = u + v$. Then $(1.9)_g$ gives rise to

$$w_t = w g^*(t, w) \tag{4.5$_{g^*}$}$$

where $g = (g^*, g^*)$, $g^* \in H(f^*)$. By [22], $(4.5)_{g^*}$ has a unique strictly positive almost-periodic solution $w(t; g^*)$ with $w(0; g^*) = u_g$. Moreover, it is not difficult to see that $\int_0^t g^*(s, w(s; g^*))\, ds$ is bounded for $t \in \mathbb{R}$ and hence is almost periodic.

Now for any $(u_0, v_0, g) \in \Gamma(g)$, $u(t; u_0, v_0, g) + v(t; u_0, v_0, g) = w(t; g^*)$. It then follows that

$$u(t; u_0, v_0, g) = u_0 e^{\int_0^t g^*(s, w(s; g^*))\, ds} \quad \text{and}$$

$$v(t; u_0, v_0, g) = v_0 e^{\int_0^t g^*(s, w(s; g^*))\, ds}.$$

This implies that $\pi_t(u_0, v_0, g)$ is almost periodic.

(2) By the arguments of Part 1, for any $(u_0, v_0, g_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$ $((u_0, v_0) \neq (0, 0))$,

$$u(t; u_0, v_0, g_0) + v(t; u_0, v_0, g_0) - w(t; g_0^*) \to 0 \quad \text{as } t \to \infty.$$

Therefore, $\omega(u_0, v_0, g_0) \cap P^{-1}(g) \subset \Gamma(g)$. By Part 1 and Theorem 4.1, if $\omega(u_0, v_0, g_0) \cap P^{-1}(g)$ is not a singleton for some $g \in H(f)$, then $\omega(u_0, v_0, g_0)$ contains two minimal sets. By the connectness of $\omega(u_0, v_0, g_0)$, there is $\tilde{g} \in H(f)$ such that $\#(\omega(u_0, v_0, g_0) \cap P^{-1}(\tilde{g})) \geq 3$. Then by Part 1 again, $\omega(u_0, v_0, g_0)$ contains at least three minimal sets, a contradiction to Theorem 4.1. Therefore, $\omega(u_0, v_0, g_0)$ is a 1-cover of $H(f)$ and hence is almost-periodic minimal. ∎

Remark 4.2. Let f be as in Corollary 4.5. Then, for any $g \in H(f)$, $(1.9)_g$ has a continuous family of positive almost-periodic solutions connecting $(u(t; u_g, 0, g), 0)$ and $(0, v(t; 0, v_g, g))$.

Given a function $h = h(t)$ which is almost periodic in t, let g_M and g_L denote $\sup_{t \in \mathbb{R}} g(t)$ and $\inf_{t \in \mathbb{R}} g(t)$, respectively.

COROLLARY 4.6. *Suppose* $f_1(t, u, v) = a_1(t) - b_1(t)u - c_1(t)v$ *and* $f_2(t, u, v) = a_2(t) - b_2(t)u - c_2(t)v$ *with* $a_{iL}, b_{iL}, c_{iL} > 0$ $(i = 1, 2)$. *Then,*

(1) *If* $a_{1L} > c_{1M}a_{2M}/c_{2L}$ *and* $a_{2L} > a_{1M}b_{2M}/b_{1L}$, *then there is a strictly positive almost-periodic minimal set* $K^* = \{(u^*(g), v^*(g), g) : g \in H(f)\}$ *of* π_t *such that for any* $(u_0, v_0, g) \in Int(\mathbb{R}^+) \in Int(\mathbb{R}^+) \times H(f)$, $\|\pi_t(u_0, v_0, g) - \pi_t(u^*(g), v^*(g), g)\| \to 0$ *as* $t \to \infty$.

(2) *If* $a_{1L} > c_{1M}a_{2M}/c_{2L}$ *and* $a_{2M} < a_{1L}b_{2L}/b_{1M}$, *then for any* $(u_0, v_0, g) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+) \times H(f)$, $\|\pi_t(u_0, v_0, g) - \pi_t(u_g, 0, g)\| \to 0$ *as* $t \to \infty$.

(3) *If* $a_{1M} < c_{1L}a_{2L}/c_{2M}$ *and* $a_{2L} > a_{1M}b_{1M}/b_{2L}$, *then for any* $(u_0, v_0, g) \in Int(\mathbb{R}^+) \times Int(\mathbb{R}^+) \times H(f)$, $\|\pi_t(u_0, v_0, g) - \pi_t(0, v_g, g)\| \to 0$ *as* $t \to \infty$.

Proof. (1) First of all, let d_1, d_2 be numbers satisfying the inequalities $d_1 > a_{1M}/b_{1L}$, $d_2 > a_{2M}/c_{1L}$, $a_{1L} - c_{1M}d_2 > 0$, and $a_{2L} - b_{2M}d_1 > 0$. Choose a number $\delta > 0$ such that

$$a_{1L} - c_{1M}d_2 - b_{1M}\delta > 0, \; a_{2L} - b_{2M}d_1 - c_{2M}\delta > 0, \text{ and } \delta < d_1, d_2.$$

Let $(u_*, v_*) = (\delta, d_2)$ and $(u^*, v^*) = (d_1, \delta)$. Then by Lemma 2.2 in [1], for any $g \in H(f)$,

$$(u_*, v_*, g \cdot t) \ll_2 \pi_t(u_*, v_*, g) \ll_2 \pi_t(u^*, v^*, g) \ll_2 (u^*, v^*, g \cdot t) \quad \text{for } t > 0. \tag{4.6}$$

Moreover, by Lemma 2.3 in [1],

$$\int_0^\infty \|\pi_t(u^*, v^*, g) - \pi_t(u_*, v_*, g)\| dt < \infty. \tag{4.7}$$

Now by Theorem 4.1 and (4.6), there is a strictly positive minimal set K of π_t such that, for any $(u, v, g) \in K$,

$$(u_*, v_*, g) \leq_2 (u, v, g) \leq_2 (u^*, v^*, g). \tag{4.8}$$

By Theorem 1 in [1], for any $(u_0, v_0, g) \in \operatorname{Int}(\mathbb{R}^+) \times \operatorname{Int}(\mathbb{R}^+) \times H(f)$ and $(u, v, g) \in K$,

$$\|\pi_t(u, v, g) - \pi_t(u_0, v_0, g)\| \to 0 \qquad \text{as } t \to \infty. \tag{4.9}$$

It is then sufficient to prove that K is almost-periodic minimal.

Assume that K is not almost-periodic minimal, and (u_1, v_1, g), $(u_2, v_2, g) \in K$ are such that $(u_1, v_1, g) <_2 (u_2, v_2, g)$, where

$$\begin{aligned} g(t, u, v) &= \big(g_1(t, u, v), g_2(t, u, v)\big) \\ &= \big(\tilde{a}_1(t) - \tilde{b}_1(t)u - \tilde{c}_1(t)v, \tilde{a}_2(t) - \tilde{b}_2(t)u - \tilde{c}_2(t)v\big). \end{aligned}$$

Let $(u_1(t), v_1(t)) = (u(t; u_1, v_1, g), v(t; u_1, v_1, g))$ and $(u_2(t), v_2(t)) = (u(t; u_2, v_2, g), v(t; u_2, v_2, g))$. By Theorem 4.1, K is an almost 1-cover of $H(f)$. Hence, there is $s_n \to -\infty$ such that $\|(u_1(s_n), v_1(s_n)) - (u_2(s_n), v_2(s_n))\| \to 0$ as $n \to \infty$. By (4.7) and (4.8),

$$\int_{s_n}^{\infty} \big(u_2(t) - u_1(t)\big)\, dt < \infty, \qquad \int_{s_n}^{\infty} \big(v_1(t) - v_2(t)\big)\, dt < \infty. \tag{4.10}$$

Note that

$$\frac{(u_i(t))_t}{u_i(t)} = \tilde{a}_1(t) - \tilde{b}_1(t)u_i(t) - \tilde{c}_1(t)v_i(t)$$

and

$$\frac{(v_i(t))_t}{v_i(t)} = \tilde{a}_2(t) - \tilde{b}_2(t)u_i(t) - \tilde{c}_2(t)v_i(t)$$

$(i = 1, 2)$. Hence,

$$\ln u_i(t) - \ln u_i(s_n) = \int_{s_n}^{t} \big(\tilde{a}_1(\tau) - \tilde{b}_1(\tau)u_i(\tau) - \tilde{c}_1(\tau)v_i(\tau)\big)\, d\tau$$

and

$$\ln v_i(t) - \ln v_i(s_n) = \int_{s_n}^{t} \big(\tilde{a}_2(\tau) - \tilde{b}_2(\tau)u_i(\tau) - \tilde{c}_2(\tau)v_i(\tau)\big)\, d\tau$$

$(i = 1, 2)$. Therefore,

$$\begin{aligned} &\ln u_1(t) - \ln u_2(t) \\ &\qquad = \ln u_1(s_n) - \ln u_2(s_n) - \int_{s_n}^{t} \tilde{b}_1(\tau)\big(u_1(\tau) - u_2(\tau)\big)\, d\tau \\ &\qquad\qquad - \int_{s_n}^{t} \tilde{c}_1(\tau)\big(v_1(\tau) - v_2(\tau)\big)\, d\tau \end{aligned}$$

and

$$\begin{aligned} &\ln v_1(t) - \ln v_2(t) \\ &\qquad = \ln v_1(s_n) - \ln v_2(s_n) - \int_{s_n}^{t} \tilde{b}_2(\tau)\big(u_1(\tau) - u_2(\tau)\big)\, d\tau \\ &\qquad\qquad - \int_{s_n}^{t} \tilde{c}_2(\tau)\big(v_1(\tau) - v_2(\tau)\big)\, d\tau. \end{aligned}$$

By (4.9), letting $t \to \infty$, we have

$$\begin{aligned} \ln u_1(s_n) - \ln u_2(s_n) &= \int_{s_n}^{\infty} \tilde{b}_1(t)\big(u_1(t) - u_2(t)\big)\, dt \\ &\quad + \int_{s_n}^{\infty} \tilde{c}_1(t)\big(v_1(t) - v_2(t)\big)\, dt \end{aligned}$$

and

$$\begin{aligned} \ln v_1(s_n) - \ln v_2(s_n) &= \int_{s_n}^{\infty} \tilde{b}_2(t)\big(u_1(t) - u_2(t)\big)\, dt \\ &\quad + \int_{s_n}^{\infty} \tilde{c}_2(t)\big(v_1(t) - v_2(t)\big)\, dt. \end{aligned}$$

This implies that

$$\begin{aligned} &\ln u_1(s_n) - \ln u_2(s_n) + b_{1L} \int_{s_n}^{\infty} \big(u_2(t) - u_1(t)\big)\, dt \\ &\qquad \le c_{1,M} \int_{s_n}^{\infty} \big(v_1(t) - v_2(t)\big)\, dt \end{aligned}$$

and

$$\begin{aligned} &\ln v_1(s_n) - \ln v_2(s_n)) + b_{2M} \int_{s_n}^{\infty} \big(u_2(t) - u_1(t)\big)\, dt \\ &\qquad \ge c_{2L} \int_{s_n}^{\infty} \big(v_1(t) - v_2(t)\big)\, dt. \end{aligned}$$

Hence,

$$\frac{\ln u_1(s_n) - \ln u_2(s_n)}{b_{1L}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} + \frac{\int_{s_n}^{\infty}(u_1(t) - u_2(t))\,dt}{\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} \leq \frac{c_{1M}}{b_{1L}}$$

and

$$\frac{\ln v_1(s_n) - \ln v_2(s_n)}{b_{2M}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} + \frac{\int_{s_n}^{\infty}(u_1(t) - u_2(t))\,dt}{\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} \geq \frac{c_{2L}}{b_{2M}}.$$

It then follows that

$$\frac{c_{1M}}{b_{1L}} \geq \frac{c_{2L}}{b_{2M}} + \frac{\ln u_1(s_n) - \ln u_2(s_n)}{b_{1L}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} - \frac{\ln v_1(s_n) - \ln v_2(s_n)}{b_{2M}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt}.$$

Note that

$$\frac{\ln u_1(s_n) - \ln u_2(s_n)}{b_{1L}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} - \frac{\ln v_1(s_n) - \ln v_2(s_n)}{b_{2M}\int_{s_n}^{\infty}(v_1(t) - v_2(t))\,dt} \to 0$$

as $n \to \infty$. We then must have

$$\frac{c_{1M}}{b_{1L}} \geq \frac{c_{2L}}{b_{2M}},$$

that is,

$$c_{1M}b_{2M} \geq b_{1L}c_{2L}.$$

But by $a_{1L} > c_{1M}a_{2M}/c_{2L}$ and $a_{2L} > a_{1M}b_{1M}/b_{1L}$, we have $b_{1L}c_{2L} > c_{1M}b_{2M}$, a contradiction. Therefore K is a 1-cover of $H(f)$, and Part 1 follows

(2) It is sufficient to prove that for any $(u_0, v_0, g_0) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+) \times H(f)$, $\omega(u_0, v_0, g_0) \cap P^{-1}(g) = (u_g, 0, g)$.

For any $g(t,u,v) = (\tilde{a}_1(t) - \tilde{b}_1(t)u - \tilde{c}_1(t)v, \tilde{a}_2(t) - \tilde{b}_2(t)u - \tilde{c}_2(t)v)$, let $(u^*, v^*, g) \in \omega(u_0, v_0, g_0) \cap P^{-1}(g)$ and $u_L = \inf_{t\in\mathbb{R}} u(t; u^*, v^*, g)$, $v_M = \sup_{t\in\mathbb{R}} v(t; u^*, v^*, g)$. Clearly, $u_L, v_M \geq 0$. We prove that $u_L > 0$ and $v_M = 0$.

First, we prove $u_L > 0$. Note that

$$\frac{a_{2L}}{c_{2M}} \le v_g \le \frac{a_{2M}}{c_{2L}}.$$

Hence, when $0 < u \ll 1$ and $|v - v_g| \ll 1$ ($v > 0$),

$$\begin{aligned} u_t &= u\big(\tilde{a}_1(t) - \tilde{b}_1(t)u - \tilde{c}_1(t)v\big) \\ &\ge u\big(a_{1L} - \tilde{b}_1(t)u - c_{1M}v\big) \\ &= u\Big(\big(a_{1L} - \tilde{b}_1(t)u - c_{1M}v_g - c_{1M}(v - v_g)\big)\Big) \\ &\ge u\left(a_{1L} - \tilde{b}_1(t)u - c_{1M}\frac{a_{2M}}{c_{2L}} - c_{1M}(v - v_g)\right). \end{aligned}$$

By $a_{1L} > c_{1M}a_{2M}/c_{2L}$, $u_t^* > 0$ for $0 < t \ll 1$. This implies that $u_L > 0$.

Next, we prove that $v_M = 0$. Assume that $v_M > 0$. Suppose t_n is such that $u^*(t_n) \to u_L$ as $n \to \infty$. Without loss of generality, assume that $v^*(t_n) \to \tilde{v}^*$, $\tilde{a}_1(t_n) \to \tilde{a}^*$, $\tilde{b}_1(t_n) \to \tilde{b}^*$, and $\tilde{c}_1(t_n) \to \tilde{c}^*$. Then we must have

$$0 \ge u_L\big(\tilde{a}_1^* - \tilde{b}_1^* u_L - \tilde{c}_1^* \tilde{v}^*\big).$$

This together with $u_L > 0$ implies that

$$a_{1L} \le b_{1M}u_L + c_{1M}v_M. \tag{4.11}$$

Similarly, by the assumption $v_M > 0$, we have

$$a_{2M} \ge b_{2L}u_L + c_{2L}v_M. \tag{4.12}$$

Multiplying (4.11) by c_{2L} and (4.12) by c_{1M} yields

$$a_{1L}c_{2L} \le b_{1M}c_{2L}u_L + c_{1M}c_{2L}v_M$$

and

$$c_{1M}a_{2M} \ge c_{1M}b_{2L}u_L + c_{1M}c_{2L}v_M.$$

It then follows that

$$a_{1L}c_{2L} - c_{1M}a_{2M} \le (b_{1M}c_{2L} - c_{1M}b_{2L})u_L. \tag{4.13}$$

Similarly, we have

$$b_{1M}a_{2M} - a_{1L}b_{2L} \ge (b_{1M}c_{2L} - c_{1M}b_{2L})v_M. \tag{4.14}$$

Therefore,

$$v_M \le \frac{b_{1M}a_{2M} - a_{1L}b_{2L}}{b_{1M}c_{2L} - c_{1M}b_{2L}}. \tag{4.15}$$

By $a_{2M} < a_{1L}b_{2L}/b_{1M}$,

$$b_{1M}a_{2M} - a_{1L}b_{2L} < 0. \tag{4.16}$$

By $a_{1L} > c_{1M}a_{2M}/c_{2L}$ and (4.13), there holds

$$b_{1M}c_{2L} - c_{1M}b_{2L} > 0. \tag{4.17}$$

Following from (4.14)–(4.17), $v_M \le 0$, a contradiction. Therefore, $v_M = 0$.

Now Lemma 3.3 together with $v_M = 0$ implies that $\omega(u_0, v_0, g_0) \cap P^{-1}(g) = (u_g, 0, g)$ for any $g \in H(f)$.

(3) This can be proved by arguments similar to those for part 2. ∎

5. CONVERGENCE TO SPATIALLY HOMOGENEOUS ALMOST PERIODIC SOLUTIONS

In this section, we study the convergence of positive solutions of (1.1) or positive motions of (1.7). We say that a set $K \subset X^+ \times X^+ \times H(k, f)$ is *spatially homogeneous* if, for any $(u, v, l, g) \in K$, (u, v) is spatially homogeneous. A positive motion $\Pi_t(u_0, v_0, l, g)$ of (1.7) is *asymptotically spatially homogeneous* if $\omega(u_0, v_0, l, g)$ is spatially homogeneous.

DEFINITION 5.1. A positive almost-periodic motion $\pi_t(u_0, v_0, g)$ of (1.10) is lower (upper) asymptotically Lyapunov stable if, for any $\epsilon > 0$, there is $\delta > 0$ such that for any $\tau \in \mathbb{R}$ and $(u, v, g \cdot \tau) \in P_2(\pi_\tau(u_0, v_0, g))$ $((u, v, g \cdot \tau) \in P_4(\pi_\tau(u_0, v_0, g)))$ with $\|(u, v, g \cdot \tau) - \pi_\tau(u_0, v_0, g)\| < \delta$, there holds

$$\|\pi_t(u, v, g \cdot \tau) - \pi_t(\pi_\tau(u_0, v_0, g))\| < \epsilon \qquad \text{for } t > 0.$$

Moreover, given $(u, v, g) \in P_2(u_0, v_0, g) \cap P_4(u_{\max}, v_{\max}, g)$ $((u, v, g) \in P_4(u_0, v_0, g) \cap P_4(u_0, v_0, g) \cap P_2(u_{\min}, v_{\min}, g))$ one has

$$\|\pi_t(u, v, g) - \pi_t(u_0, v_0, g)\| \to 0 \qquad \text{as } t \to \infty,$$

where $(u_{\max}, v_{\max}, g)$ $((u_{\min}, v_{\min}, g))$ is the maximum (minimum) element satisfying that

$$(u_{\max}, v_{\max}, g) <_2 (u_0, v_0, g) \qquad ((u_0, v_0, g) <_2 (u_{\min}, v_{\min}, g))$$

and $\pi_t(u_{\max}, v_{\max}, g)$ ($\pi_t(u_{\min}, v_{\min}, g)$) is a positive almost-periodic motion.

Remark 5.1. The set $P_2(u_0, v_0, g) \cap P_4(u_{\max}, v_{\max}, g)$ ($P_4(u_0, v_0, g) \cap P_2(u_{\min}, v_{\min}, g)$) in Definition 5.1 may be empty, and if it is empty then $\pi_t(u_0, v_0, g)$ is just lower (upper) Lyapunov stable in the usual sense.

DEFINITION 5.2. A positive almost-periodic minimal set K of π_t is globally stable if, for any $(u, v, g) \in \mathrm{Int}(\mathbb{R}^+) \times \mathrm{Int}(\mathbb{R}^+) \times H(f)$,

$$\|\pi_t(u, v, g) - \pi_t(u_0, v_0, g)\| \to 0 \qquad \text{as } t \to \infty,$$

where $\{(u_0, v_0, g)\} = K \cap P^{-1}(g)$.

Throughout this section we shall use the abbreviations (H1)–(H4) for the following assumptions.

(H1) $\omega(u_0, v_0, g)$ is almost-periodic minimal for all $(u_0, v_0, g) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$.

(H2) Each strictly positive almost-periodic motion of (1.10) (if one exists) is lower asymptotically Lyapunov stable. Moreover, either $\pi_t(u_g, 0, g)$ is lower asymptotically Lyapunov stable for all $g \in H(f)$ or it is unstable for all $g \in H(f)$, and it is lower asymptotically Lyapunov stable if there is no strictly positive almost-periodic motions of (1.10).

(H3) Each strictly positive almost-periodic motion of (1.10) (if one exists) is upper asymptotically Lyapunov stable. Moreover, either $\pi_t(0, v_g, g)$ is upper asymptotically Lyapunov stable for all $g \in H(f)$ or it is unstable for all $g \in H(f)$, and it is upper asymptotically Lyapunov stable if there is no strictly positive almost-periodic motion of (1.10).

(H4) Equation (1.10) has a globally stable positive almost-periodic minimal set.

Remark 5.2. (1) Assume (H1) and (H2). If (1.10) has a unique strictly positive almost-periodic minimal set K (hence, it has a unique strictly positive almost-periodic motion for each $g \in H(f)$) and if $\pi_t(u_g, 0, g)$ is unstable for all $g \in H(f)$, then K is globally stable. If (1.10) has no strictly positive almost-periodic motions, then $K = \{(u_g, 0, g)\colon g \in H(f)\}$ is globally stable.

(2) Assume (H1) and (H3). If (1.10) has a unique strictly positive almost-periodic minimal set K (hence, it has a unique strictly positive almost-periodic motion for each $g \in H(f)$) and $\pi_t(0, v_g, g)$ is unstable for all $g \in H(f)$, then K is globally stable. If (1.10) has no strictly positive almost-periodic motions, then $K = \{(0, v_g, g)\colon g \in H(f)\}$ is globally stable.

Now, the main result of this section can be stated as follows.

THEOREM 5.1. *Assume* (H1) *and* (H2) *or* (H3) *or* (H4). *Then for any* $(u_0, v_0, l_0, g_0) \in X^+ \times X^+ \times H(k, f)$, $\omega(u_0, v_0, l_0, g_0)$ *is a* 1-*cover of* $H(f)$ *and is spatially homogeneous. Hence,* $\Pi_t(u_0, v_0, l_0, g_0)$ *is asymptotically almost periodic and spatially homogeneous.*

The theorem will be proved after the following lemma.

LEMMA 5.2. *Consider* $(1.9)_g$.

(1) *Assume* (H1) *and* (H2). *Suppose that* $\pi_t(u_1, v_1, g)$ *is a strictly positive almost-periodic motion of* π_t *and* $(u, v, g) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$ *with* $(u_1, v_1, g) <_2 (u, v, g)$. *If there is a strictly positive motion* $\pi_t(u_2, v_2, g)$ *with*

$$(u_1, v_1, g) <_2 (u_2, v_2, g) \text{ and } \omega(u_1, v_1, g) <_2 \omega(u_2, v_2, g), \quad (5.1)$$

then $\omega(u_1, v_1, g) <_2 \omega(u, v, g)$.

(2) *Assume* (H1) *and* (H3). *Suppose that* $\pi_t(u_1, v_1, g)$ *is a strictly positive almost-periodic motion of* π_t *and* $(u, v, g) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H(f)$ *with* $(u, v, g) <_2 (u_1, v_1, g)$. *If there is a strictly positive motion* $\pi_t(u_2, v_2, g)$ *with*

$$(u_2, v_2, g) <_2 (u_1, v_1, g) \text{ and } \omega(u_2, v_2, g) <_2 \omega(u_1, v_1, g), \quad (5.2)$$

then $\omega(u, v, g) <_2 \omega(u_1, v_1, g)$.

Proof. (1) By (5.1) and (H1), either $\omega(u_2, v_2, g)$ is strictly positive or $\omega(u_2, v_2, g) \cap P^{-1}(g) = (u_g, 0, g)$ and $\pi_t(u_g, 0, g)$ is lower asymptotically Lyapunov stable. Now by (5.1) and (H2), $\omega(u_1, v_1, g) <_2 \omega(u, v, g)$.

(2) It can be proved by arguments similar to those for Part 1. ∎

Proof of Theorem 5.1. Without loss of generality, we may assume that $(u_0, v_0) \in \operatorname{Int}(X^+) \times \operatorname{Int}(X^+)$.

Case 1. Equation (1.2) satisfies (H1) and (H2).

First, we prove that any minimal set $K \subset \omega(u_0, v_0, l_0, g_0)$ is spatially homogeneous and hence is a 1-cover of $H(k, f)$. Suppose that $(u, v, l, g) \in K$ is not spatially homogeneous. Let $u_1 = \min_{x \in \Omega} u(x)$, $v_1 = \max_{x \in \Omega} v(x)$, $u_2 = \max_{x \in \Omega} u(x)$, and $v_2 = \min_{x \in \Omega} v(x)$. Then we have

$$(u_1, v_1) <_2 (u, v) <_2 (u_2, v_2).$$

Moreover, for any $(u_*, v_*) <_2 (u, v)$ and $(u, v) <_2 (u^*, v^*)$, we have $(u_*, v_*) \leq_2 (u_1, v_1)$ and $(u_2, v_2) \leq_2 (u^*, v^*)$. By Lemma 3.2,

$$\Pi_t(u_1, v_1, l, g) \ll_2 \Pi_t(u, v, l, g) \ll_2 \Pi_t(u_2, v_2, l, g) \qquad \text{for } t > 0. \quad (5.3)$$

Fix $t_0 > 0$. By (5.3), there are $(\tilde{u}_1, \tilde{v}_1, l \cdot t_0, g \cdot t_0)$ and $(\tilde{u}_2, \tilde{v}_2, l \cdot t_0, g \cdot t_0)$ with $\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2 \in \operatorname{Int} \mathbb{R}^+$ such that

$$\Pi_{t_0}(u_1, v_1, l, g) \ll_2 \left(\tilde{u}_1, \tilde{v}_2, l \cdot t_0, g \cdot t_0\right) <_2 \Pi_{t_0}(u, v, g)$$
$$<_2 \left(\tilde{u}_2, \tilde{v}_2, l \cdot t_0, g \cdot t_0\right) \ll_2 \Pi_{t_0}(u_2, v_2, l, g).$$

Hence

$$\Pi_{t+t_0}(u_1, v_1, l, g) \ll_2 \Pi_t\left(\tilde{u}_1, \tilde{v}_1, l \cdot t_0, g \cdot t_0\right) \ll_2 \Pi_{t+t_0}(u, v, l, g)$$
$$\ll_2 \Pi_t\left(\tilde{u}_2, \tilde{v}_2, l \cdot t_0, g \cdot t_0\right) \ll_2 \Pi_{t+t_0}(u_2, v_2, l, g) \quad (5.4)$$

for $t > 0$. Let $t_n \to \infty$ be such that

$$\Pi_{t_n+t_0}(u, v, l, g) \to (u, v, l, g).$$

Without loss of generality, assume that the limits $(u_i^*, v_i^*, l, g) = \lim_{n\to\infty} \Pi_{t_n+t_0}(u_i, v_i, l, g)$ and $(\tilde{u}_i^*, \tilde{v}_i^*, l, g) = \lim_{n\to\infty} \Pi_{t_n}(\tilde{u}_i, \tilde{v}_i, l \cdot t_0, g \cdot t_0)$ exist $(i = 1, 2)$. Then

$$(u_1^*, v_1^*, l, g) \le_2 \left(\tilde{u}_1^*, \tilde{v}_1^*, l, g\right) <_2 (u, v, l, g) <_2 \left(\tilde{u}_2^*, \tilde{v}_2^*, l, g\right)$$
$$\le_2 (u_2^*, v_2^*, l, g).$$

Hence,

$$(u_1^*, v_1^*, l, g) \le_2 \left(\tilde{u}_1^*, \tilde{v}_1^*, l, g\right) \le_2 (u_1, v_1, l, g) <_2 \left(\tilde{u}_2^*, \tilde{v}_2^*, l, g\right). \quad (5.5)$$

Now by (5.4), (5.5), and Lemma 5.2,

$$\begin{aligned}(u_1^*, v_1^*, l, g) &= \lim_{n\to\infty} \Pi_{t_n+t_0}(u_1, v_1, l, g) <_2 \lim_{n\to\infty} \Pi_{t_n}\left(\tilde{u}_1, \tilde{v}_1, l \cdot t_0, g \cdot t_0\right)\\ &= \left(\tilde{u}_1^*, \tilde{v}_1^*, l, g\right)\\ &= \lim_{n\to\infty} \Pi_{t_n+t_0}\left(\tilde{u}_1^*, \tilde{v}_1^*, l, g\right) \le_2 \lim_{n\to\infty} \Pi_{t_n+t_0}(u_1, v_1, l, g)\\ &= (u_1^*, v_1^*, l, g),\end{aligned}$$

a contradiction. Hence, K is spatially homogeneous.

Next, we prove that $\omega(u_0, v_0, l_0, g_0)$ contains only one minimal set. Suppose that $K_1, K_2 \subset \omega(u_0, v_0, l_0, g_0)$ are two distinct minimal sets. Then both are spatially homogeneous and almost periodic. Therefore, K_1 and K_2 are ordered with respect to $\le_2$. Let $(u_1, v_1, l_0, g_0) \in K_1$ and $(u_2, v_2, l_0, g_0) \in K_2$ and assume that $(u_1, v_1, l_0, g_0) \ll_2 (u_2, v_2, l_0, g_0)$. Not that there is $t_n \to \infty$ such that

$$\Pi_{t_n}(u_0, v_0, l_0, g_0) \to (u_2, v_2, l_0, g_0) \qquad \text{as } n \to \infty. \quad (5.6)$$

Hence there is $T > 0$ such that $\pi_T(u_1, v_1, l_0, g_0) \ll_2 \pi_T(u_0, v_0, l_0, g_0)$. Let

$$(u_0^*, v_0^*) = \left(\max_{x\in\Omega} u_0(x), \min_{x\in\Omega} v_0(x)\right).$$

Then

$$\pi_T(u_1, v_1, l_0, g_0) \ll_2 \Pi_T(u_0, v_0, l_0, g_0) \leq_2 \Pi_T(u_0^*, v_0^*, l_0, g_0). \quad (5.7)$$

By (5.6), (5.7), and (H1),

$$K_1 <_2 \omega(u_0^*, v_0^*, l_0, g_0). \quad (5.8)$$

Note that $\pi_t(u_0^*, v_0^*, g_0)$ is strictly positive. It then follows from (5.7), (5.8), and Lemma 5.2 that $K_1 <_2 \omega(u_0, v_0, l_0, g_0)$, a contradiction. Therefore, $\omega(u_0, v_0, l_0, g_0)$ contains only one minimal set.

Finally, we prove that $\omega(u_0, v_0, l_0, g_0)$ is minimal. Suppose that $K \subset \omega(u_0, v_0, l_0, g_0)$ is minimal and $(u_1, v_1, l, g) \in K$, $(u_2, v_2, l, g) \in \omega(u_0, v_0, l_0, g_0) \setminus K$. Then there is $s_n \to -\infty$ such that

$$\pi_{s_n}(u_i, v_i, l, g) \to (u_1, v_1, l, g) \ (i = 1, 2) \qquad \text{as } n \to \infty. \quad (5.9)$$

Note that $(u_2, v_2, l, g) \notin (\{0\} \times X^+ \times H(k, f)) \cup (X^+ \times \{0\} \times H(k, f))$. By (5.9) and (H2), $(u_1, v_1, l, g) \neq (0, v_g, l, g)$ and, for any $\epsilon > 0$, $(u_1 - \epsilon, v_1 + \epsilon, l, g) <_2 (u_2, v_2, l, g)$. Hence, $(u_1, v_1, l, g) <_2 (u_2, v_2, l, g)$. Without loss of generality, we may assume that $(u_1, v_1, l, g) \ll_2 (u_2, v_2, l, g)$. Note also that there is $t_n \to \infty$ such that

$$\begin{aligned} &\Pi_{t_n}(u_0, v_0, l_0, g_0) \to (u_2, v_2, l, g) \qquad \text{and} \\ &\Pi_{t_n}(u_1^*, v_1^*, l_0, g_0) \to (u_1, v_1, l, g) \end{aligned} \quad (5.10)$$

as $n \to \infty$, where $(u_1^*, v_1^*, l_0, g_0) \in K$. Then there is $T > 0$ such that

$$\Pi_T(u_1^*, v_1^*, l_0, g_0) \ll_2 \Pi_T(u_0, v_0, l_0, g_0).$$

As above, let $(u_0^*, v_0^*) = (\max_{x \in \Omega} u_0(x), \min_{x \in \Omega} v_0(x)$. Then

$$\Pi_T(u_1^*, v_1^*, l_0, g_0) \ll_2 \Pi_T(u_0, v_0, l_0, g_0) \leq_2 \Pi_T(u_0^*, v_0^*, l_0, g_0). \quad (5.11)$$

It then follows from (5.10) and (5.11) that

$$K <_2 \omega(u_0^*, v_0^*, l_0, g_0). \quad (5.12)$$

By (5.11), (5.12), and Lemma 5.2, $K <_2 \omega(u_0, v_0, l_0, g_0)$, a contradiction. Therefore, $\omega(u_0, v_0, l_0, g_0)$ is minimal and the theorem follows.

Case 2. Equation (1.2) satisfies (H1) and (H3). By arguments similar to those for Case 1, the theorem holds.

Case 3. Equation (1.2) satisfies (H1) and (H4).

Let $K = \{(u^g, v^g, g) : g \in H(f)\}$ be the globally stable positive almost-periodic minimal set of π_t. Let $u_1 = \min_{x \in \bar{\Omega}} u_0(x)$, $v_1 = \max_{x \in \bar{\Omega}} v_0(x)$, $u_2 = \max_{x \in \bar{\Omega}} u_0(x)$, and $v_2 = \min_{x \in \bar{\Omega}} v_0(x)$. Then (u_1, v_1), (u_2, v_2) are strictly positive and

$$(u_1, v_1, l_0, g_0) \leq_2 (u_0, v_0, l_0, g_0) \leq_2 (u_2, v_2, l_0, g_0).$$

By Lemma 3.2,

$$\Pi_t(u_1, v_1, l_0, g_0) \leq_2 \Pi_t(u_0, v_0, l_0, g_0) \leq_2 \Pi_t(u_2, v_2, l_0, g_0) \quad (5.13)$$

for $t > 0$. By (H4), $\omega(u_1, v_1, l_0, g_0) = \omega(u_2, v_2, l_0, g_0) = K$. This and (5.13) imply that $\omega(u_0, v_0, l_0, g_0) = K$. The theorem follows. ∎

COROLLARY 5.3. *Let f be as in Corollary* 4.5. *Then for any* $(u_0, v_0, l_0, g_0) \in X^+ \times X^+ \times H(k, f)$, $\omega(u_0, v_0, l_0, g_0)$ *is almost-periodic minimal and is spatially homogeneous.*

Proof. By Lemma 3.6 and the arguments of Corollary 4.5, (1.2) satisfies (H1) and (H2). The corollary then follows from Theorem 5.1. ∎

COROLLARY 5.4. *Let f be as in Corollary* 4.6. *Then*

(1) *If* $a_{1L} > c_{1M} a_{2M}/c_{2L}$ *and* $a_{2L} > a_{1M} b_{2M}/b_{1L}$, *then there is a unique spatially homogeneous strictly positive almost-periodic minimal set* $K^* = \{(u^*(g), v^*(g), l, g) : (l, g) \in H(k, f)\}$ *of* Π_t *such that for any* $(u_0, v_0, l, g) \in \mathit{Int}(X^+) \times \mathit{Int}(X^+) \times H(k, f)$,

$$\|\Pi_t(u_0, v_0, l, g) - \Pi_t(u^*(g), v^*(g), l, g)\| \to 0 \qquad \text{as } t \to \infty.$$

(2) *If* $a_{1L} > c_{1M} a_{2M}/c_{2L}$ *and* $a_{2M} < a_{1L} b_{2L}/b_{1M}$, *then for any* $(u_0, v_0, l, g) \in \mathit{Int}(X^+) \times \mathit{Int}(X^+) \times H(k, f)$, $\|\Pi_t(u_0, v_0, l, g) - \Pi_t(u_g, 0, l, g)\| \to 0$ *as* $t \to \infty$.

(3) *If* $a_{1M} < c_{1L} a_{2L}/c_{2M}$ *and* $a_{2L} > a_{1M} b_{2M}/b_{1L}$, *then for any* $(u_0, v_0, l, g) \in \mathit{Int}(X^+) \times \mathit{Int}(X^+) \times H(k, f)$, $\|\Pi_t(u_0, v_0, l, g) - \Pi_t(0, v_g, l, g)\| \to 0$ *as* $t \to \infty$.

Proof. By Corollary 4.6, (1.2) satisfies (H1) and (H4). The corollary then follows from the arguments of Theorem 5.1. ∎

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